THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 3: Conditional probability and Bayes rule

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Example: Umbrella sales

Anita works for an umbrella company. She gets a bonus if and only if she sells more than 10 umbrellas in a day.

- If it is raining, the probability that she sells more than 10 umbrellas is 0.8.

- If it isn't raining, the probability that she sells more than 10 umbrellas is 0.25.

- The probability that it rains tomorrow is 0.1.

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  - $P(R) = 0.1$.

What is the probability that it doesn't rain tomorrow, and she gets her bonus? What is $P(R^c \cap B)$?

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We can rearrange our formula for conditional probability. Since $P(W|R^c) = \frac{P(W \cap R^c)}{P(R^c)}$, we also have:

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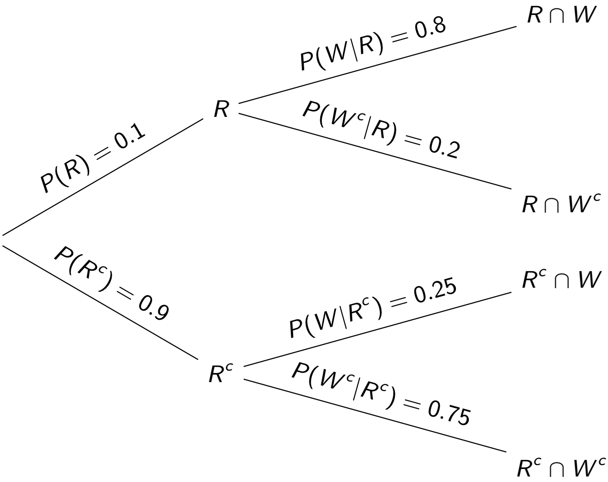
- But I wanted $P(B \cap R^c)$! Isn't this the same as $P(W \cap R^c)$?

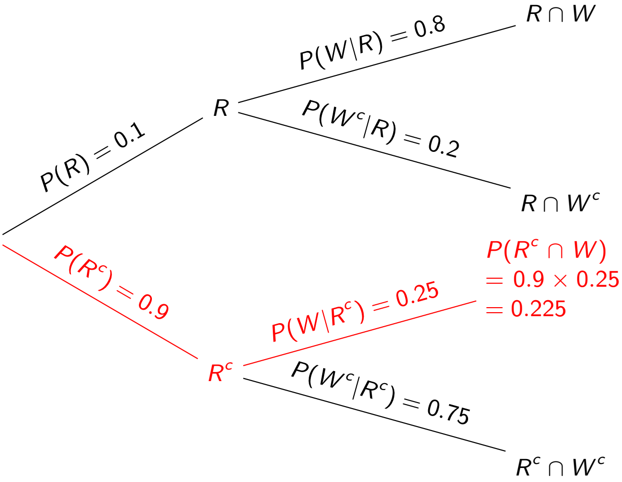
Representing conditional probabilities using a tree

We can represent conditional probabilities using a tree structure.



Representing conditional probabilities using a tree

The probability at a leaf node is the product of the probabilities along each path.



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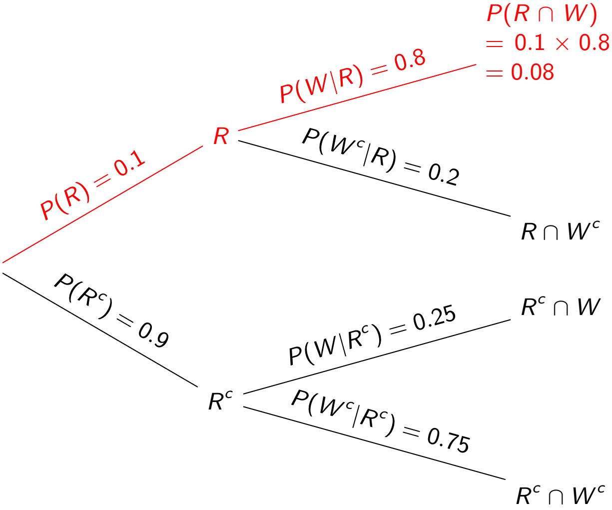

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- But $P(R) = P(B \cap C) = P(B|C)P(C)$.
- Using induction you can prove that:

$$P(\cap_{i=1}^{n} A_i) = P(A_1|A_2 \cap \cdots \cap A_n)P(A_2|A_3 \cap \cdots \cap A_n) \cdots P(A_{n-1}|A_n)P(A_n)$$

Revisit Example: Umbrella sales

Anita works for an umbrella company. She gets a bonus (event B) iff she sells more than 10 umbrellas in a day (event W).

- Now $W = B$. We have $P(R) = 0.1$, $P(W|R) = 0.8$ and $P(W|R^c) = 0.25$.
- If you knew that Anita got a bonus, then what is the probability that it rained?

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  - Now additivity gives

$$\begin{aligned} P(W) &= P(W \cap R) + P(W \cap R^c) \\ &= P(W|R)P(R) + P(W|R^c)P(R^c) = 0.08 + 0.25 \times 0.9 \approx 0.3 \end{aligned}$$

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The last step is also known as **Bayes Rule**, which we will study next.

Multiplication rule: examples

Three cards are drawn from an ordinary 52 card deck without replacement. What is the probability that there is no heart among the three?

- Without replacement: drawn cards are not placed back into the deck.
- Notation: $A_i = \{i^{th}\text{card is not a heart}\}$
- Remember: There are thirteen cards with hearts.

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- Use multiplication rule:
  $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$.
- $P(A_1) =$. $P(A_2|A_1) =$. $P(A_3|A_1 \cap A_2) =$.

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- $P(A_1) = \frac{39}{52}$. $P(A_2|A_1) =$. $P(A_3|A_1 \cap A_2) =$.

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- $P(A_1) = \frac{39}{52}$. $P(A_2|A_1) = \frac{38}{51}$. $P(A_3|A_1 \cap A_2) = \frac{37}{50}$.

Multiplication rule: examples

I have two black balls, and one red ball. I pick two balls randomly without replacement.

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- What is the probability that the second ball is black?

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P(X_2 = B) &= P(X_2 = B \cap X_1 = B) + P(X_2 = B \cap X_1 = R) \\
&= P(X_2 = B|X_1 = B)P(X_1 = B) + P(X_2 = B|X_1 = R)P(X_1 = R) \\
&= \frac{1}{2} \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{2}{3}
\end{aligned}$$

- Is this obvious? If you know nothing about the first ball, then the second ball can be any one of the 3 balls you have.

Reading

- Read Sections 1.1, 1.2 and 1.3 of Bertsekas and Tsitsiklis.