THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 19: Continuous random variables-Derived distributions, max of two independent r.v.'s

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Roadmap

- Two random variables: joint distributions
  - Joint pdf ✓
  - Joint pdf to a single pdf: Marginalization ✓
  - Conditional pdf
    - Conditioning on an event ✓
    - Conditioning on a continuous r.v ✓
    - Total probability rule for continuous r.v's ✓
    - Bayes theorem for continuous r.v's ✓
    - Conditional expectation and total expectation theorem✓
  - Independence ✓
- More than two random variables. ✓
- Derived distributions
  - Linear functions
  - Monotonic functions

Try it yourself: linear function of continuous random variables

- Let X be a random variable with PDF $f_X(x)$, and let $Y = 2X + 3$
- Then the CDF of Y is given by:

$$\begin{aligned} F_Y(y) &= P(2X + 3 \leq y) = P\left(X \leq \frac{y-3}{2}\right) \\ &= F_X\left(\frac{y-3}{2}\right) \end{aligned}$$

- Differentiating, we get:

$$\begin{aligned} f_Y(y) &= \frac{dF_Y(y)}{dy} = \frac{dF_X\left(\frac{y-3}{2}\right)}{dy} \\ &= \frac{1}{2} f_X\left(\frac{y-3}{2}\right) \end{aligned}$$

Try it yourself: linear function of continuous random variables

- Let X be a random variable with PDF $f_X(x)$, and let $Y = -2X + 3$
- Then the CDF of Y is given by:

$$\begin{aligned} F_Y(y) &= P(-2X + 3 \leq y) = P\left(X \geq \frac{3-y}{2}\right) \\ &= 1 - F_X\left(\frac{3-y}{2}\right) \end{aligned}$$

- Differentiating, we get:

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Functions of continuous random variables

- Let X be a random variable with PDF $f_X(x)$, and let $Y = aX + b$
-

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a > 0 \\ -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Linear functions of continuous random variables

- So, for any continuous random variable X, if $Y = aX + b$, then

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- Let's consider a normal random variable... $X \sim N(0,1)$, then what is the distribution of $Y = aX + b$?

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- So,

$$\begin{aligned} f_Y(y) =& \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|}\frac{1}{\sqrt{2\pi}} \exp\{-\left(\frac{y-b}{a}\right)^2/2\} \\ =& \frac{1}{\sqrt{2\pi}|a|} \exp\left(-\frac{(y-b)^2}{2a^2}\right) \end{aligned}$$

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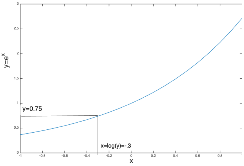
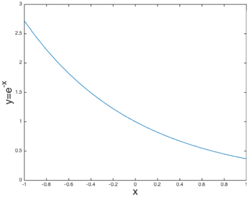
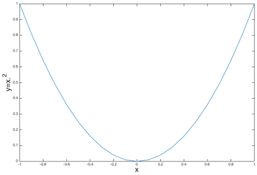
- This is just the PDF of a normal distribution with mean b and variance a^2!
- In fact, if $X \sim N(\mu, \sigma^2)$, then $Y = aX + b \sim N(a\mu + b, a^2\sigma^2)$.

How about strictly monotonic functions?

- Another special case is where g is a strictly monotonic function of X.
- Let X be a continuous random variable whose PDF is non-zero only in some range I. A function $g(X)$ is said to be strictly monotonic over I if it is either:
  - **Monotonically increasing**: if $x < x'$ then $g(x) < g(x')$ for all $x, x' \in I$.
  - **Monotonically decreasing**: if $x < x'$ then $g(x) > g(x')$ for all $x, x' \in I$.
  - In the following picture which one is which?







Monotonically increasing

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- We have $Y = e^X$. What is $f_Y(y)$?
- Start with the CDF .

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$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y) = F_X(\ln y)$$

- $$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(\ln y)}{dy} = \frac{1}{y} f_X(\ln y)$$

Monotonically decreasing

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- We have $Y = e^{-X}$. What is $f_Y(y)$?
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Monotonic functions of continuous random variables

- Any strictly monotonic function can be inverted, that is there exists an **inverse** function h (also known as g^{-1}) such that

$$g(x) = y \qquad \text{if and only if} \qquad x = h(y) \qquad \text{For all } x \in I$$

- If I tell you that $g(x) = ax + b$, then what is the inverse function?

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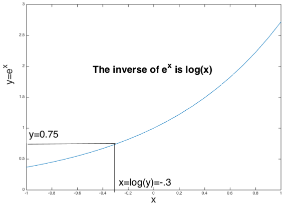
- 

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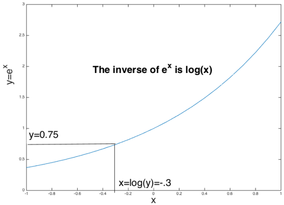
    - 

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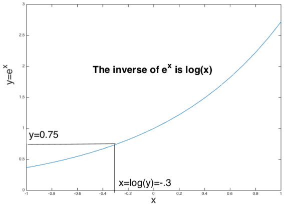

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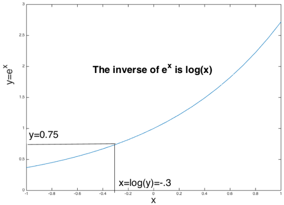

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- If g is monotonically increasing, so is h. If g is monotonically decreasing, so is h.

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- We can differentiate $F_Y(y)$ to get

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(h(y))\frac{dh(y)}{dy}$$

- Since h is monotonically increasing, it's slope, $\frac{dh(y)}{dy}$, must be nonnegative.

Monotonic functions of continuous random variables

- Similarly we can show that when h is monotonically decreasing,

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(h(y))\left(-\frac{dh(y)}{dy}\right)$$

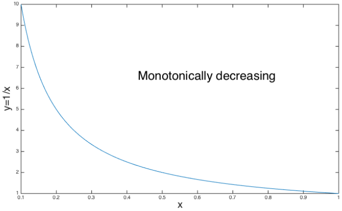
- Since h is monotonically decreasing, it's slope, $\dfrac{dh}{dy}(y)$, must be negative.

- So, for general strictly monotonic functions of X, we have

$$\boxed{f_Y(y) = f_X(h(y))\left|\frac{dh(y)}{dy}\right|}$$

Example: Monotonically decreasing function

- Let X be a uniform random variable on the interval $(0, 1]$, and let $Y = g(X) = 1/X$.
- So Y takes values in $[1, \infty)$



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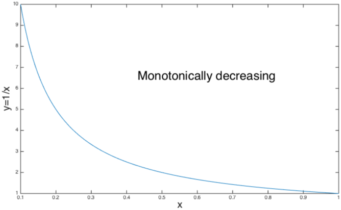

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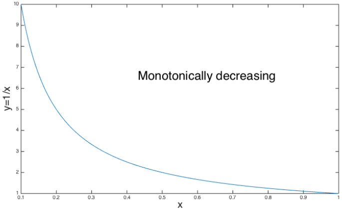

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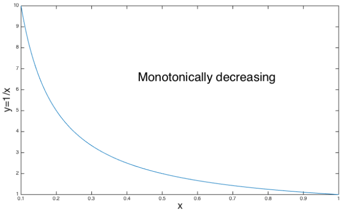

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- So for $y \geq 1$, $f_X(h(y)) = f_X(1/y) = 1$.

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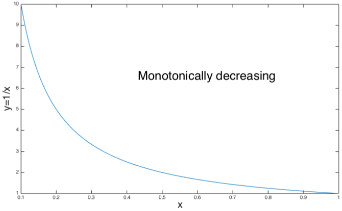

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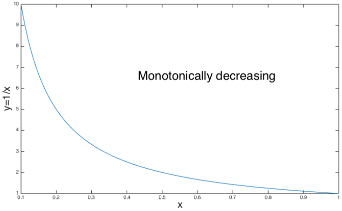

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- $\left|\frac{dh(y)}{dy}\right| = \left|-\frac{1}{y^2}\right| = \frac{1}{y^2}$
- So,

$$f_Y(y) = f_X(h(y)) \left|\frac{dh}{dy}(y)\right| = \begin{cases} \frac{1}{y^2} & y \geq 1 \\ 0 & y < 1 \end{cases}$$

Functions of a continuous random variable: Summary

- We know that functions $Y = g(X)$ of random variables X are again random variables.
- In the discrete case, it was pretty easy to find the PMF of the new random variable X:

$$p_Y(y) = \sum_{x|g(x)=y} p_X(x)$$

- In the continuous case, we need to find $F_Y(y) = P(Y \leq y) = \int_{x|g(X)\leq y} f_X(x)dx$
- We can then differentiate wrt y to get the PDF of Y.
- In certain cases, this procedure simplifies a little:
- **Linear case**: If $Y = aX + b$, $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$
- **Monotonic case**: If g is a strictly monotonic function with inverse h, then $f_Y(y) = f_X(h(y))\left|\frac{dh}{dy}(y)\right|$
- What about if we have functions of more than one random variable?

Functions of two random variables

- If X and Y are both random variables, then $Z = g(X, Y)$ is also a random variable.
- In the discrete case, we could easily find the PMF of the new random variable:

$$p_Z(z) = \sum_{x,y|g(x,y)=z} p_{X,Y}(x, y)$$

- For example, if I roll two fair dice, what is the probability that the sum is 6?

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- For example, if I roll two fair dice, what is the probability that the sum is 6?
- Each possible ordered pair has probability $1/36$.
- The options that sum to 6 are $(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$... or in other words $(k, 6 - k)$ for $k = 1, \ldots, 5$.
- So, $p_Z(5) = 5/36$

Functions of two independent continuous random variables

- I know two guests will both arrive in the next hour. The arrival times are independent random variables which follow a uniform distribution. What is the PDF of the arrival time of the last guest to arrive?
- $X \sim \mathit{Uniform}([0, 1])$ and $Y \sim \mathit{Uniform}([0, 1])$.
- In other words, what is the PDF of $Z = \max(X, Y)$?

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- Differentiating, $f_Z(z) = \begin{cases} 2z & 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Functions of two random variables: Summary

- If $Y = g(X)$, in order to get the PDF of Y we first looked at the CDF, $P(Y \leq y) = P(g(X) \leq y)$ and then differentiated with respect to y.
- For functions $Z = g(X, Y)$ of two random variables, the same general idea applies:
- First we look at the CDF, $P(Z \leq z) = P(g(X, Y) \leq x)$
- Then we differentiate with respect to z.
- We looked at a special case: The maximum of two independent r.v.'s.
- This procedure is known as convolution.