THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 20: Practice problems

Purnamrita Sarkar
Department of Statistics and Data Science
The University of Texas at Austin

www.cs.cmu.edu/∼psarkar/teaching

Continuous r.v

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- If $E[X] = \frac{3}{5}$, find a and b.

Continuous r.v

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- If $E[X] = \frac{3}{5}$, find a and b.
  - Normalize: $\int_{x=0}^{1} (a + bx^2) dx = 1$.

Continuous r.v

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- If $E[X] = \frac{3}{5}$, find a and b.
  - Normalize: $\int_{x=0}^{1} (a + bx^2)dx = 1$. Solving $a + b/3 = 1$

Continuous r.v

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- If $E[X] = \frac{3}{5}$, find a and b.
  - Normalize: $\int_{x=0}^{1} (a + bx^2) dx = 1$. Solving $a + b/3 = 1$
  - $E[X] = \int_0^1 x(a + bx^2) dx = 3/5$. Solving $a/2 + b/4 = 3/5$, i.e. $a + b/2 = 6/5$

Continuous r.v

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- If $E[X] = \frac{3}{5}$, find a and b.
  - Normalize: $\int_{x=0}^{1} (a + bx^2)dx = 1$. Solving $a + b/3 = 1$
  - $E[X] = \int_0^1 x(a + bx^2)dx = 3/5$. Solving $a/2 + b/4 = 3/5$, i.e. $a + b/2 = 6/5$
  - Solving the two equations we get: $a = 3/5, b = 6/5$.

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- What is $\text{var}(X)$?

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- What is var(X)?
  - $E[X^2] = \int_0^1 x^2(a + bx^2)dx = a/3 + b/5 = 1/5 + 6/25 = 11/25$
  - $\text{var}(X) = E[X^2] - E[X]^2 = 11/25 - (3/5)^2 = 2/25$

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- What is $\text{var}(X)$?
  - $E[X^2] = \int_0^1 x^2(a + bx^2)dx = a/3 + b/5 = 1/5 + 6/25 = 11/25$
  - $\text{var}(X) = E[X^2] - E[X]^2 = 11/25 - (3/5)^2 = 2/25$
- $Y = X - 1$. What is $E[XY]$?

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- What is var(X)?
  - $E[X^2] = \int_0^1 x^2(a + bx^2)dx = a/3 + b/5 = 1/5 + 6/25 = 11/25$
  - $\text{var}(X) = E[X^2] - E[X]^2 = 11/25 - (3/5)^2 = 2/25$

- $Y = X - 1$. What is $E[XY]$?
  - $E[XY] = E[X]E[Y] = 3/5(3/5 - 1) = -6/25$. Is this correct?

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- What is var(X)?
  - $E[X^2] = \int_0^1 x^2(a + bx^2)dx = a/3 + b/5 = 1/5 + 6/25 = 11/25$
  - $\text{var}(X) = E[X^2] - E[X]^2 = 11/25 - (3/5)^2 = 2/25$

- $Y = X - 1$. What is $E[XY]$?
  - $E[XY] = E[X]E[Y] = 3/5(3/5 - 1) = -6/25$. Is this correct? No! This is only true is X, Y are independent. Are they?

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- What is $\text{var}(X)$?
  - $E[X^2] = \int_0^1 x^2(a + bx^2)dx = a/3 + b/5 = 1/5 + 6/25 = 11/25$
  - $\text{var}(X) = E[X^2] - E[X]^2 = 11/25 - (3/5)^2 = 2/25$

- $Y = X - 1$. What is $E[XY]$?
  - $E[XY] = E[X]E[Y] = 3/5(3/5 - 1) = -6/25$. Is this correct? No! This is only true is X, Y are independent. Are they?
  - $E[XY] = E[X(X - 1)] = E[X^2] - E[X] = 11/25 - 3/5 = -4/5$

Continuous r.v

A continuous r.v. X has density function

$$f(x) = c\, e^{-|x|} = \begin{cases} c\, e^{-x} & x \geq 0 \\ c\, e^{x} & x < 0 \end{cases}$$

1. Find c.
   - We have
     $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} ce^{x}dx + \int_{0}^{\infty} ce^{-x}dx =$$
2. Find $p(|X| > 2)$.

3. Find $E(X)$ (*Hint:* no integral required).

Continuous r.v

A continuous r.v. X has density function

$$f(x) = c\, e^{-|x|} = \begin{cases} c\, e^{-x} & x \geq 0 \\ c\, e^{x} & x < 0 \end{cases}$$

1. Find c.
   - We have
   $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} ce^{x}dx + \int_{0}^{\infty} ce^{-x}dx = 2c = 1.c = 1/2$$
2. Find $p(|X| > 2)$.

3. Find $E(X)$ (*Hint:* no integral required).

Continuous r.v

A continuous r.v. X has density function

$$f(x) = c\, e^{-|x|} = \begin{cases} c\, e^{-x} & x \geq 0 \\ c\, e^{x} & x < 0 \end{cases}$$

1. Find c.
   - We have $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} ce^{x}dx + \int_{0}^{\infty} ce^{-x}dx = 2c = 1.c = 1/2$
2. Find $p(|X| > 2)$.
   - $P(|X| > 2) = P(X < -2) + P(X > 2) = 2 \times 0.5 \displaystyle\int_{-\infty}^{-2} e^{x}dx = e^{-2}.$
3. Find $E(X)$ (*Hint:* no integral required).

Continuous r.v

A continuous r.v. X has density function

$$f(x) = c\,e^{-|x|} = \begin{cases} c\,e^{-x} & x \geq 0 \\ c\,e^{x} & x < 0 \end{cases}$$

1. Find c.
   - We have $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} ce^{x}dx + \int_{0}^{\infty} ce^{-x}dx = 2c = 1.c = 1/2$
2. Find $p(|X| > 2)$.
   - $P(|X| > 2) = P(X < -2) + P(X > 2) = 2 \times 0.5 \int_{-\infty}^{-2} e^{x}dx = e^{-2}.$
3. Find $E(X)$ (*Hint:* no integral required).
   - $E[X] = 0$. Why?

Continuous r.v

A continuous r.v. X has density function

$$f(x) = c\,e^{-|x|} = \begin{cases} c\,e^{-x} & x \geq 0 \\ c\,e^{x} & x < 0 \end{cases}$$

1. Find c.
   - We have $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} ce^{x}dx + \int_{0}^{\infty} ce^{-x}dx = 2c = 1.c = 1/2$
2. Find $p(|X| > 2)$.
   - $P(|X| > 2) = P(X < -2) + P(X > 2) = 2 \times 0.5 \int_{-\infty}^{-2} e^{x}dx = e^{-2}.$
3. Find $E(X)$ (*Hint:* no integral required).
   - $E[X] = 0$. Why?
   - **For any r.v. X with a PDF symmetric around 0,** $E[X] = 0$

Useful tricks to go around integration by parts

- You have to calculate $\int_0^\infty ve^{-v}dv$, but you don't remember integration by parts.
- Remember that the expectation of $X \sim Exp(\lambda)$ is

$$E[X] = \lambda \int_0^\infty xe^{-\lambda x}dx = \frac{1}{\lambda}$$

Useful tricks to go around integration by parts

- You have to calculate $\int_0^\infty ve^{-v}dv$, but you don't remember integration by parts.
- Remember that the expectation of $X \sim Exp(\lambda)$ is

$$E[X] = \lambda \int_0^\infty xe^{-\lambda x}dx = \frac{1}{\lambda}$$

- So by comparison, $\int_0^\infty ve^{-v}dv$ is the expectation of $Y \sim Exp(1)$ and is 1.

Useful tricks to go around integration by parts

- You have to calculate $\int_0^\infty ve^{-v}dv$, but you don't remember integration by parts.
- Remember that the expectation of $X \sim Exp(\lambda)$ is

$$E[X] = \lambda \int_0^\infty xe^{-\lambda x}dx = \frac{1}{\lambda}$$

- So by comparison, $\int_0^\infty ve^{-v}dv$ is the expectation of $Y \sim Exp(1)$ and is 1.
- Can you get $\int_0^\infty 3ve^{-2v}dv$?

Useful tricks to go around integration by parts

- You have to calculate $\int_0^\infty ve^{-v}dv$, but you don't remember integration by parts.
- Remember that the expectation of $X \sim Exp(\lambda)$ is

$$E[X] = \lambda \int_0^\infty xe^{-\lambda x}dx = \frac{1}{\lambda}$$

- So by comparison, $\int_0^\infty ve^{-v}dv$ is the expectation of $Y \sim Exp(1)$ and is 1.
- Can you get $\int_0^\infty 3ve^{-2v}dv$?
- $$\int_0^\infty 3ve^{-2v}dv = 3\int_0^\infty ve^{-2v}dv = 1.5 \times \int_0^\infty (2v)e^{-2v}dv = 1.5E[Y] = .75$$
  where $Y \sim Exp(2)$.

Useful tricks to go around integration by parts

- You have to calculate $\int_0^\infty v^2 e^{-v/2} dv$.
- Remember that the variance of $X \sim Exp(\lambda)$ is

$$E[X^2] = \lambda \int_0^\infty x^2 exp(-\lambda x) dx = \text{var}(X) + 1/\lambda^2 = \frac{2}{\lambda^2}$$

- So $\int_0^\infty v^2 e^{-v/2} dv = 2 \int_0^\infty 1/2 v^2 e^{-v/2} dv$ is just $2E[X^2]$ where $X \sim Exp(1/2)$. So it equals 16.

Useful tricks to go around integration by parts

- You have to calculate $\int_0^\infty e^{-v^2/2} dv$.
- Whats the distribution that you should compare with?

Useful tricks to go around integration by parts

- You have to calculate $\int_0^\infty e^{-v^2/2} dv$.
- Whats the distribution that you should compare with?
- $\int_0^\infty e^{-x^2/2} dx = P(X \geq 0) = 1/2$, where $X \sim N(0,1)$.
- $\int_0^\infty e^{-v^2/2} dv = \sqrt{2\pi} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-v^2/2} dv = \sqrt{2\pi} P(X \geq 0) = \sqrt{\pi/2}$

Continuous r.v

You have $X \sim N(0,1)$.

- What is $E[X|X \geq 0]$?
  - First, what is $f_{X|X\geq 0}(x) = f_X(x)/P(X \geq 0)$
  - But what is $P(X \geq 0)$?
  - So $E[X|X \geq 0] = \int_0^\infty x f_{X|X\geq 0}(x)dx = 2\int_0^\infty x f_X(x)dx = \frac{2}{\sqrt{2\pi}}\int_0^\infty x e^{-x^2/2}dx$. You could integrate this.
  - Let $v = x^2/2$. Now $E[X|X \geq 0] = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-v}dv = \sqrt{2/\pi}$
- What is $E[X|X < 0]$
  - You could use the same calculation. OR....

Continuous r.v

You have $X \sim N(0,1)$.

- What is $E[X|X \geq 0]$?
  - First, what is $f_{X|X\geq 0}(x) = f_X(x)/P(X \geq 0)$
  - But what is $P(X \geq 0)$? 1/2 by symmetry.
  - So $E[X|X \geq 0] = \int_0^\infty x f_{X|X\geq 0}(x)dx = 2\int_0^\infty x f_X(x)dx = \frac{2}{\sqrt{2\pi}}\int_0^\infty xe^{-x^2/2}dx$. You could integrate this.
  - Let $v = x^2/2$. Now $E[X|X \geq 0] = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-v}dv = \sqrt{2/\pi}$
- What is $E[X|X < 0]$
  - You could use the same calculation. OR....
  - Use total expectation theorem!

Continuous r.v

You have $X \sim N(0,1)$.

- What is $E[X|X \geq 0]$?
  - First, what is $f_{X|X\geq 0}(x) = f_X(x)/P(X \geq 0)$
  - But what is $P(X \geq 0)$? 1/2 by symmetry.
  - So $E[X|X \geq 0] = \int_0^\infty x f_{X|X\geq 0}(x)dx = 2\int_0^\infty x f_X(x)dx = \frac{2}{\sqrt{2\pi}}\int_0^\infty xe^{-x^2/2}dx$. You could integrate this.
  - Let $v = x^2/2$. Now $E[X|X \geq 0] = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-v}dv = \sqrt{2/\pi}$
- What is $E[X|X < 0]$
  - You could use the same calculation. OR....
  - Use total expectation theorem!
  - $E[X] = E[X|X \geq 0]P(X \geq 0) + E[X|X < 0]P(X < 0)$. But $E[X] =$?

Continuous r.v

You have $X \sim N(0,1)$.

- What is $E[X|X \geq 0]$?
  - First, what is $f_{X|X\geq 0}(x) = f_X(x)/P(X \geq 0)$
  - But what is $P(X \geq 0)$? 1/2 by symmetry.
  - So $E[X|X \geq 0] = \int_0^\infty x f_{X|X\geq 0}(x)dx = 2\int_0^\infty x f_X(x)dx = \frac{2}{\sqrt{2\pi}}\int_0^\infty xe^{-x^2/2}dx$. You could integrate this.
  - Let $v = x^2/2$. Now $E[X|X \geq 0] = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-v}dv = \sqrt{2/\pi}$
- What is $E[X|X < 0]$
  - You could use the same calculation. OR....
  - Use total expectation theorem!
  - $E[X] = E[X|X \geq 0]P(X \geq 0) + E[X|X < 0]P(X < 0)$. But $E[X] = ?0$
  - $E[X|X \geq 0] \times 1/2 + E[X|X < 0] \times 1/2 = 0$. So, $E[X|X < 0] = -E[X|X \geq 0] = -\sqrt{2/\pi}$

Uniform r.v's

A man and a woman agree to meet at a certain location at about 12:30 p.m. Suppose the man arrives at a time uniformly distributed between 12:00 and 12:45, and the woman independently arrives at a time uniformly distributed between 12:15 and 1:00. Let X be the man's arrival time and Y be the woman's arrival time.

- What is the joint PDF?
- What is the probability that the man arrives first?

Uniform r.v's

Find the PDF first.

$$f_X(x) = \begin{cases} 1/45 & x \in [12:00, 12:45] \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1/45 & y \in [12:15, 1:00] \\ 0 & \text{otherwise} \end{cases}$$

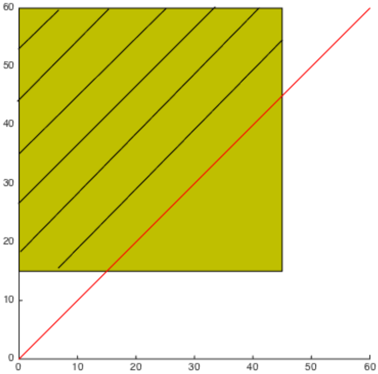
Using independence,

$$f_{X,Y}(x,y) = \begin{cases} 1/45^2 & x \in [12:00, 12:45], y \in [12:15, 1] \\ 0 & \text{otherwise} \end{cases}$$

Uniform r.v's

Find the probability that the man arrives first.

$$P(X < Y) = 1 - 1/45^2 \times .5 \times 30^2 = 1 - \frac{2}{9} = 7/9$$



Conditional expectation

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- $P(X > 8 | X > 4) =$

Conditional expectation

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- $P(X > 8 | X > 4) = P(X > 8, X > 4)/P(X > 4) =$

Conditional expectation

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- $P(X > 8 | X > 4) = P(X > 8, X > 4)/P(X > 4) = P(X > 8)/P(X > 4) = e^{-8\lambda}/e^{-4\lambda} = e^{-4\lambda}$

Conditional expectation

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- $P(X > 8|X > 4) = P(X > 8, X > 4)/P(X > 4) = P(X > 8)/P(X > 4) = e^{-8\lambda}/e^{-4\lambda} = e^{-4\lambda}$
- In general, $P(X > s + t|X > s) = P(X > t)$. Memoryless property, just like the geometric.

Conditional expectation

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- $P(X > 8|X > 4) = P(X > 8, X > 4)/P(X > 4) = P(X > 8)/P(X > 4) = e^{-8\lambda}/e^{-4\lambda} = e^{-4\lambda}$
- In general, $P(X > s + t|X > s) = P(X > t)$. Memoryless property, just like the geometric.
- $E[X] = \lambda \int_0^\infty e^{-\lambda x} dx = \int_0^\infty ue^{-u} du/\lambda = 1/\lambda$

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- $P(X > 8 | X > 4) =$

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim$ exponential(λ).

- $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- $P(X > 8 | X > 4) = P(X > 8, X > 4)/P(X > 4) =$

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim$ exponential(λ).

- $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- $P(X > 8 | X > 4) = P(X > 8, X > 4)/P(X > 4) = P(X > 8)/P(X > 4) = e^{-8\lambda}/e^{-4\lambda} = e^{-4\lambda}$

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim$ exponential(λ).

- $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- $P(X > 8 | X > 4) = P(X > 8, X > 4)/P(X > 4) = P(X > 8)/P(X > 4) = e^{-8\lambda}/e^{-4\lambda} = e^{-4\lambda}$
- In general, $P(X > s + t | X > s) = P(X > t)$. Memoryless property, just like the geometric.

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- $P(X > 8 | X > 4) = P(X > 8, X > 4)/P(X > 4) = P(X > 8)/P(X > 4) = e^{-8\lambda}/e^{-4\lambda} = e^{-4\lambda}$
- In general, $P(X > s + t | X > s) = P(X > t)$. Memoryless property, just like the geometric.
- $E[X] = \lambda \int_0^\infty e^{-\lambda x} dx = \int_0^\infty u e^{-u} du/\lambda = 1/\lambda$

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim$ exponential(λ).

- What about $E[X|X > 4]$? Well,
  $f_{X|X>4}(x) = f_X(x)/P(X > 4) = \lambda e^{-\lambda(x-4)}$ for $x \geq 4$

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- What about $E[X|X > 4]$? Well, $f_{X|X>4}(x) = f_X(x)/P(X > 4) = \lambda e^{-\lambda(x-4)}$ for $x \geq 4$
- But this is an exponential distribution again! Memoryless property.

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- What about $E[X|X > 4]$? Well, $f_{X|X>4}(x) = f_X(x)/P(X > 4) = \lambda e^{-\lambda(x-4)}$ for $x \geq 4$
- But this is an exponential distribution again! Memoryless property.
- So $E[X|X > 4] = 4 + E[X] = 4 + 1/\lambda = 6$

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- What about $E[X|X>4]$? Well, $f_{X|X>4}(x) = f_X(x)/P(X>4) = \lambda e^{-\lambda(x-4)}$ for $x \geq 4$
- But this is an exponential distribution again! Memoryless property.
- So $E[X|X>4] = 4 + E[X] = 4 + 1/\lambda = 6$
- $\lambda \int_4^\infty x e^{-\lambda(x-4)} dx = \lambda \int_0^\infty (4+y) e^{-\lambda y} dy = 4 \int_0^\infty (\lambda e^{-\lambda y}) dy + \int_0^\infty y(\lambda e^{-\lambda y}) dy = 4 + E[Y] = 4 + 1/\lambda$

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- What about $E[X|X > 4]$? Well, $f_{X|X>4}(x) = f_X(x)/P(X > 4) = \lambda e^{-\lambda(x-4)}$ for $x \geq 4$
- But this is an exponential distribution again! Memoryless property.
- So $E[X|X > 4] = 4 + E[X] = 4 + 1/\lambda = 6$
- $\lambda \int_4^\infty x e^{-\lambda(x-4)} dx = \lambda \int_0^\infty (4 + y) e^{-\lambda y} dy = 4 \int_0^\infty (\lambda e^{-\lambda y}) dy + \int_0^\infty y(\lambda e^{-\lambda y}) dy = 4 + E[Y] = 4 + 1/\lambda$
- How do you calculate $E[X|X \leq 4]$?

Exponential r.v.

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- What about $E[X|X > 4]$? Well, $f_{X|X>4}(x) = f_X(x)/P(X > 4) = \lambda e^{-\lambda(x-4)}$ for $x \geq 4$
- But this is an exponential distribution again! Memoryless property.
- So $E[X|X > 4] = 4 + E[X] = 4 + 1/\lambda = 6$
- $\lambda \int_4^\infty x e^{-\lambda(x-4)} dx = \lambda \int_0^\infty (4+y) e^{-\lambda y} dy = 4 \int_0^\infty (\lambda e^{-\lambda y}) dy + \int_0^\infty y(\lambda e^{-\lambda y}) dy = 4 + E[Y] = 4 + 1/\lambda$
- How do you calculate $E[X|X \leq 4]$?
- Remember! $E[X] = E[X|X \leq 4]P(X \leq 4) + E[X|X > 4]P(X > 4)$. So $2 = 1/\lambda = E[X|X \leq 4](1 - e^{-4}) + (4 + 1/\lambda)e^{-4}$. So $E[X|X \leq 4] = \dfrac{2 - 6e^{-4}}{1 - e^{-4}}$

Correlation: Example of $|\rho| = 1$

- We throw a biased coin, with probability of heads p, n times. Let X be the number of heads, and let Y be the number of tails.
- $Y = n - X$
- $E[X] =$

Correlation: Example of $|\rho| = 1$

- We throw a biased coin, with probability of heads p, n times. Let X be the number of heads, and let Y be the number of tails.
- $Y = n - X$
- $E[X] = np$, and $E[Y] = n(1 - p) = n - E[X]$.
- $\text{var}(X) = np(1 - p) = \text{var}(Y)$.

Correlation: Example of $|\rho| = 1$

- We throw a biased coin, with probability of heads p, n times. Let X be the number of heads, and let Y be the number of tails.
- $Y = n - X$
- $E[X] = np$, and $E[Y] = n(1 - p) = n - E[X]$.
- $\text{var}(X) = np(1 - p) = \text{var}(Y)$.
- $\text{cov}(X, Y) = \text{cov}(X, n - X) = \text{cov}(X, -X) = -\text{cov}(X, X) = -\text{var}(X)$

Correlation: Example of $|\rho| = 1$

- We throw a biased coin, with probability of heads p, n times. Let X be the number of heads, and let Y be the number of tails.
- $Y = n - X$
- $E[X] = np$, and $E[Y] = n(1-p) = n - E[X]$.
- $\text{var}(X) = np(1-p) = \text{var}(Y)$.
- $\text{cov}(X, Y) = \text{cov}(X, n - X) = \text{cov}(X, -X) = -\text{cov}(X, X) = -\text{var}(X)$
- The correlation coefficient is therefore

$$\rho_{X,Y} = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{-\text{var}(X)}{\sqrt{\text{var}(X)\text{var}(X)}} = -1$$

Correlation: Example of $|\rho| = 1$

- We throw a biased coin, with probability of heads p, n times. Let X be the number of heads, and let Y be the number of tails.
- $Y = n - X$
- $E[X] = np$, and $E[Y] = n(1-p) = n - E[X]$.
- $\text{var}(X) = np(1-p) = \text{var}(Y)$.
- $\text{cov}(X, Y) = \text{cov}(X, n - X) = \text{cov}(X, -X) = -\text{cov}(X, X) = -\text{var}(X)$
- The correlation coefficient is therefore

$$\rho_{X,Y} = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{-\text{var}(X)}{\sqrt{\text{var}(X)\text{var}(X)}} = -1$$

Correlation: Example of $|\rho| = 1$

- We throw a biased coin, with probability of heads p, n times. Let X be the number of heads, and let Y be the number of tails.
- $Y = n - X$
- $E[X] = np$, and $E[Y] = n(1-p) = n - E[X]$.
- $\text{var}(X) = np(1-p) = \text{var}(Y)$.
- $\text{cov}(X, Y) = \text{cov}(X, n - X) = \text{cov}(X, -X) = -\text{cov}(X, X) = -\text{var}(X)$
- The correlation coefficient is therefore

$$\rho_{X,Y} = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{-\text{var}(X)}{\sqrt{\text{var}(X)\text{var}(X)}} = -1$$

Correlation: Example of $|\rho| = 1$

- We throw a biased coin, with probability of heads p, n times. Let X be the number of heads, and let Y be the number of tails.
- $Y = n - X$
- $E[X] = np$, and $E[Y] = n(1-p) = n - E[X]$.
- $\text{var}(X) = np(1-p) = \text{var}(Y)$.
- $\text{cov}(X, Y) = \text{cov}(X, n - X) = \text{cov}(X, -X) = -\text{cov}(X, X) = -\text{var}(X)$
- The correlation coefficient is therefore

$$\rho_{X,Y} = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{-\text{var}(X)}{\sqrt{\text{var}(X)\text{var}(X)}} = -1$$

- Remember $X = n - Y$, so they have a linear relationship.

Correlation: Example of $|\rho| = 1$

- Let $Y = aX + b$.

Correlation: Example of $|\rho| = 1$

- Let $Y = aX + b$.
-

$$\text{cov}(X, aX + b) = \text{cov}(X, aX) = a\text{var}(X)$$

$$\rho_{X,aX+b} = \frac{\text{cov}(X, aX)}{\sqrt{\text{var}(X)a^2\text{var}(X)}} = \frac{a\text{var}(X)}{|a|\text{var}(X)} = \frac{a}{|a|}$$

$$= \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases}$$

More conditional expectation

$$f_X(x) = \begin{cases} 1 & x \in [-1/2, 0] \\ e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Compute $E[X|X > 0]$.
- Compute $E[X|X \leq 0]$.
- Compute $E[X]$.

More conditional expectation

$$f_X(x) = \begin{cases} 1 & x \in [-1/2, 0] \\ e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- What is $f_{X|X>0}(x)$?

$$f_{X|X>0}(x) = \begin{cases} e^{-2x}/P(X > 0) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- $P(X > 0) = 1 - P(X \leq 0) = 1/2$.

$$f_{X|X>0}(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- So conditioned on $X > 0$, $X \sim \mathit{Exponential}(2)$. So $E[X|X > 0] = 1/2$.

More conditional expectation

$$f_X(x) = \begin{cases} 1 & x \in [-1/2, 0] \\ e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- What is $f_{X|X \leq 0}(x)$?

$$f_{X|X \leq 0}(x) = \begin{cases} 1/P(X < 0) = 2 & x \in [-1/2, 0] \\ 0 & \text{otherwise} \end{cases}$$

- So conditioned on $X \leq 0$, $X \sim \mathit{Uniform}([-1/2, 0])$. So $E[X|X < 0] = (-1/2 + 0)/2 = -1/4$.

Total expection theorem

$$f_X(x) = \begin{cases} 1 & x \in [-1/2, 0] \\ e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- What is $E[X]$?

$$\begin{aligned} E[X] &= E[X|X > 0]P(X > 0) + E[X|X \leq 0]P(X \leq 0) \\ &= 1/4 + (-1/4)1/2 = 1/8 \end{aligned}$$