THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

# SDS 321: Introduction to Probability and Statistics

## Lecture 2: Conditional probability

Purnamrita Sarkar
Department of Statistics and Data Science
The University of Texas at Austin

www.cs.cmu.edu/∼psarkar/teaching

# Example

Alice and Bob toss two fair coins separately. Denote the event that Alice gets a $H$ by $E_A$ and the event that Bob gets a $H$ by $E_B$. We know that $P(E_A) = P(E_B) = 1/2$.

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- So the union is really $P(E_A \cup E_B) = P(E_A) + P(E_B) - P(E_A \cap E_B) = 3/4$.
- May be what you are really thinking about is independent and not disjoint events.

# Partial information

- So far, we have assumed we know nothing about the outcome of our experiment, except for the information encoded in the probability law.
- Sometimes, however, we have **partial information** that may affect the likelihood of a given event.
  - The experiment involves rolling a die. You are told that the number is odd.
  - The experiment involves the weather tomorrow. You know that the weather today is rainy.
  - The experiment involves the presence or absence of a disease. A blood test comes back positive.
- In each case, knowing about some event $B$ (e.g. "it is raining today") changes our beliefs about event $A$ ("Will it rain tomorrow?").
- We want to **update** our probability law to incorporate this new knowledge.

# Conditional Probability

**Original problem**:

- What is the probability of some event $A$.
  - e.g. What is the probability that we roll a number less than 4?
- In other words, what is $P(A)$?
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**New problem**:

- Assuming event $B$ (equivalently given event $B$), what is the probability of event $A$?
  - e.g. Given that the number rolled is an odd number, what is the probability that it is less than 4?
- We call this the **conditional distribution** of $A$ given $B$.
- We write this as $P(A|B)$
- Read $|$ as "given" or "conditioned on the fact that".
- Our conditional probability is still describing "the probability of something", so we expect it to behave like a probability distribution.

# Conditional Probability

- Consider rolling a fair 6-sided die (uniform, discrete probability distribution).
- Let $A$ be the event "outcome is equal to 1'.
  - What is $P(A)$?
- Let's now assume that the number rolled is an odd number.
  - What is the set, $B$, that we are conditioning on?
- What do you think $P(A|B)$ should be?

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- More generally, $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.
- A conditional probability is only defined if $P(B) > 0$.

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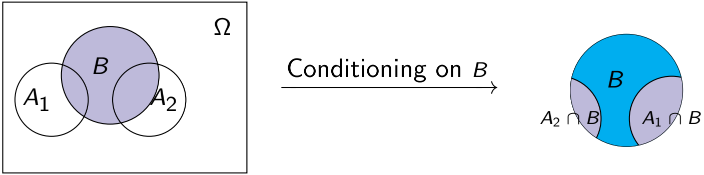

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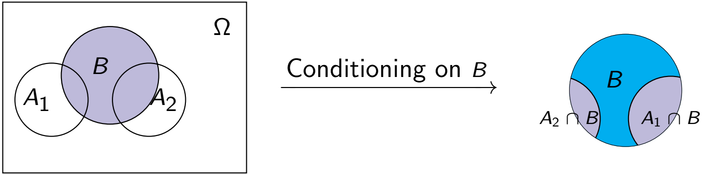


Using additivity, $P((A_1 \cup A_2) \cap B) = P(A_1 \cap B) + P(A_2 \cap B)$, so

$$P(A_1 \cup A_2|B) = \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} = P(A_1|B) + P(A_2|B)$$

# Properties of conditional probability

If $P(B) > 0$,

- If $A_i$ for $i \in \{1, \ldots, n\}$ are all pairwise disjoint, then
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# Example: Coin toss

Consider the experiment of tossing a fair coin three times. What is the probability of getting alternating heads and tails conditioned on the event that your first toss gives a head?

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  - So, $P(A|B) = 1/4$.

# Homework

- Read Sections 1.1, 1.2 and 1.3 of Bertsekas and Tsitsiklis.
- The first homework will be posted online today. It is due next Thursday by 5pm via Canvas.