THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 19: Continuous random variables: Independence, covariance

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Roadmap

- Independence
- Covariance and correlation.
- More than two random variables.
- Function of random variables

Independent random variables-example

- You have two random variables X, Y with joint PDF

$$f_{XY}(x, y) = \begin{cases} cxy & x, y \in [0, 1] \\ 0 & \text{Otherwise} \end{cases}$$

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- $f_Y(y) = \begin{cases} 2y & y \in [0,1] \\ 0 & \text{Otherwise} \end{cases}$
- $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x, y \in [0,1]$.

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- $f_Y(y) = \begin{cases} \int_y^1 8xydx = 4y(1-y^2) & y \in [0,1] \\ 0 & \text{Otherwise} \end{cases}$
- $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$ for all $x, y \in [0,1]$.

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- You have two random variables X, Y with joint PDF

$$f_{XY}(x, y) = \begin{cases} ce^{-2x}e^{-3y} & x, y \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

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- Compute $E[XY]$.

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- Next note that $e^{-2x}e^{-3y}$ is basically the product of a function of x and a function of y. If I gave you $f_{X,Y}(x,y) = c(x+y)$, then its not a product of a function of x and a function of y. But in the given problem, X, Y are independent.

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- Next look at the two bits, for x we have e^{-2x}. So this is sort of like an exponential. So $f_X(x) = 2e^{-2x}$ and $f_Y(y) = 3e^{-3x}$.

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- So $E[X] = 1/2$ and $E[Y] = 1/3$ and via independence $E[XY] = 1/6$

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- You have two random variables X, Y with joint PDF

$$f_{XY}(x,y) = \begin{cases} ce^{-2x}e^{-y^2/2} & x \geq 0, -\infty \leq y \leq \infty \\ 0 & \text{Otherwise} \end{cases}$$

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- $f_X(x) = 2e^{-2x}$, i.e. $X \sim \mathit{Exponential}(2)$
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- $c = 2/\sqrt{2\pi}$.
- $E[X] = 1/2$ and $E[Y] = 0$.

Covariance

- The **covariance** of two random variables X and Y is given by

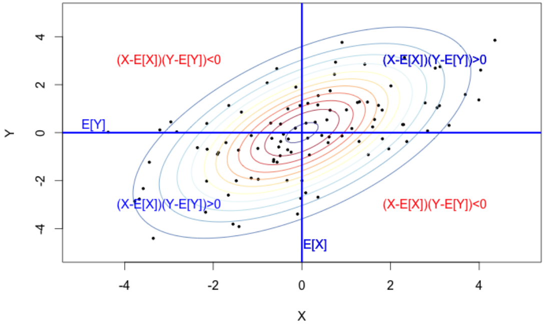
$$\text{cov}(X, Y) = E\left[(X - E[X])(Y - E[Y])\right]$$

- We can simplify this a little

$$\begin{aligned}\text{cov}(X, Y) =& E[(X - E[X])(Y - E[Y])] \\ =& E[XY - XE[Y] - YE[X] + E[X]E[Y]] \\ =& E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ =& E[XY] - E[X][E[Y]\end{aligned}$$

- It is a measure of how much X and Y change together.
- A **positive** covariance means that, if $X > E[X]$, we are likely to have $Y > E[Y]$
- A **negative** covariance means that, if $X > E[X]$, we are likely to have $Y < E[Y]$.

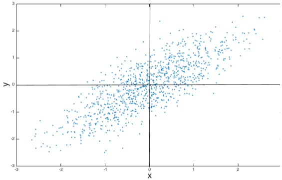
Covariance

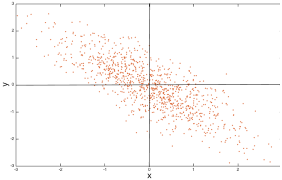


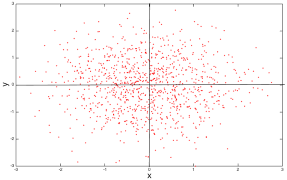
- A **positive** covariance means that we have most mass in the upper right and lower left quadrants.
- A **negative** covariance means that we have most mass in the upper left and lower right quadrants.
- A **zero** covariance means that we have about an equal mass in the upper left and upper right quadrants.

Covariance

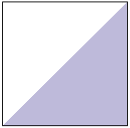
- We are plotting two random variables X and Y below. Which one corresponds to a positive, negative or zero covariance?







Example: Continuous case



- Let $f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$
- What is $\text{cov}(X, Y)$?

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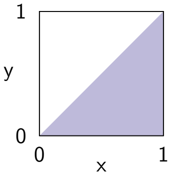

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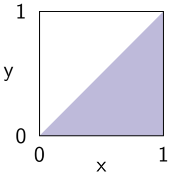

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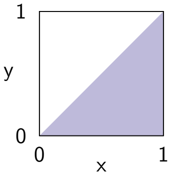

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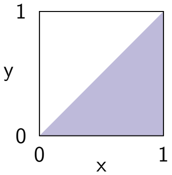

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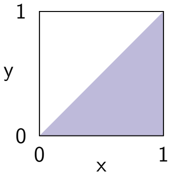

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- So the expectations are:

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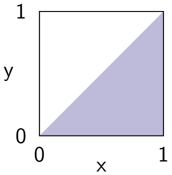

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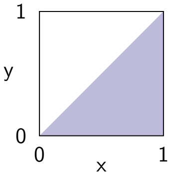

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- So the expectations are:

$$E[X] = \int_0^1 2x^2 dx = 2/3$$

$$E[Y] = \int_0^1 (2y - 2y^2)dy = 1/3$$

Example: Continuous case

- We next need to calculate $E[XY]$.
- This is just the expectation of a function of two random variables

$$E[XY] = \int_0^1 \int_0^1 xy f_{X,Y}(x,y)\,dx\,dy$$

Example: Continuous case

- We next need to calculate $E[XY]$.
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$$
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E[XY] &= \int_0^1 \int_0^1 xyf_{X,Y}(x,y)dxdy \\
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&= \int_{x=0}^1 x\left[y^2\right]_0^x dx = \int_{x=0}^1 x^3 dx = 1/4
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\end{aligned}
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- So, $\text{cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{4} - \frac{1}{3}\cdot\frac{2}{3} = \frac{1}{36}$

Covariance and Independence

- If two random variables are **independent**, knowing one tells us nothing about the other!
- In this case, $E[XY] = E[X]E[Y]$
- We know that $\text{cov}(X, Y) = E[XY] - E[X]E[Y]$... so if two random variables are independent, their covariance is zero.
- This shouldn't be surprising... we know X can't tell us anything about Y.
- What about the converse? If $\text{cov}(X, Y) = 0$, does that mean that X and Y are independent?

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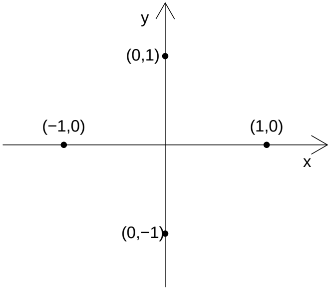
- What about the converse? If $\text{cov}(X, Y) = 0$, does that mean that X and Y are independent?
- Another way of asking this is, does $E[XY] = E[X]E[Y]$ imply X and Y are independent?

Covariance and Independence

- I start at co-ordinates (0,0). I pick a compass direction (N,S,E,W) uniformly at random, and walk 1 unit in that direction.
- Let (X, Y) be my new coordinates. My sample space is $\{(0,1),(1,0),(0,-1),(-1,0)\}$.



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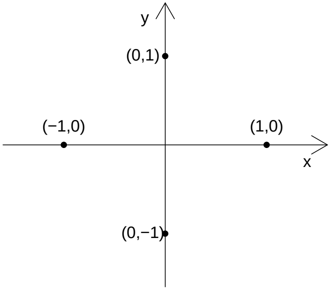

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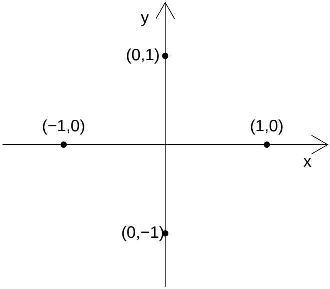

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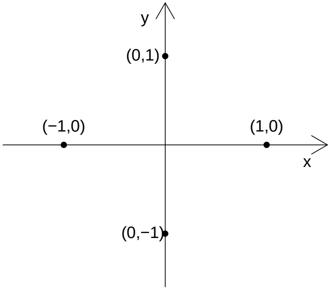
Covariance and Independence

- I start at co-ordinates (0,0). I pick a compass direction (N,S,E,W) uniformly at random, and walk 1 unit in that direction.
- Let (X, Y) be my new coordinates. My sample space is $\{(0,1),(1,0),(0,-1),(-1,0)\}$.



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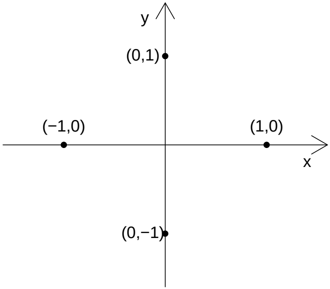

- What are $E[X]$ and $E[Y]$? 0.
- $XY = 0$.
- So, $\text{cov}(X, Y) = 0$.
- But, if I know $X = 1$, then I *must* have $Y = 0$. So, they are not independent!

Independence implies zero correlation... but zero correlation does not imply independence!

Properties of covariance

- $\text{cov}(X, Y) = E[XY] - E[X]E[Y] = \text{cov}(Y, X)$

Properties of covariance

- What is the covariance of X and X?

Properties of covariance

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- $\text{cov}(X, X) = \text{var}(X)$

Properties of covariance

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Properties of covariance

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- $$\begin{aligned} \text{cov}(X, a) &= E[Xa] - E[X]E[a] \\ &= aE[X] - aE[X] = 0 \end{aligned}$$

Properties of covariance

- What is the covariance of X and aY?

Properties of covariance

- What is the covariance of X and aY?
- $\text{cov}(X, aY) = E[aXY] - E[X]E[aY] = a\text{cov}(X, Y)$

Properties of covariance

- What is the covariance of X and $Y + b$?

Properties of covariance

- What is the covariance of X and $Y + b$?
$$\text{cov}(X, Y + b) = E[X(Y + b)] - E[X]E[Y + b]$$
- $$= E[XY] + bE[X] - (E[X]E[Y] + bE[X])$$
$$= \text{cov}(X, Y)$$

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- What is the covariance of $X_1 + X_2$ and $Y_1 + Y_2$?
- $$\begin{aligned}\text{cov}(X_1 + X_2, Y_1 + Y_2) &= E[(X_1 + X_2)(Y_1 + Y_2)] - E[X_1 + X_2]E[Y_1 + Y_2] \\ &= E[X_1 Y_1] + E[X_1 Y_2] + E[X_2 Y_1] + E[X_2 Y_2] \\ &\quad - (E[X_1]E[Y_1] + E[X_1]E[Y_2] + E[X_2]E[Y_1] + E[X_2]E[Y_2]) \\ &= (E[X_1 Y_1] - E[X_1]E[Y_1]) + (E[X_1 Y_2] - E[X_1]E[Y_2]) \\ &\quad + (E[X_2 Y_1] - E[X_2]E[Y_1]) + (E[X_2 Y_2] - E[X_2]E[Y_2]) \\ &= \text{cov}(X_1, Y_1) + \text{cov}(X_1, Y_2) + \text{cov}(X_2, Y_1) + \text{cov}(X_2, Y_2)\end{aligned}$$

Properties of covariance

- What is the covariance of $\sum_{i=1}^{n} a_i X_i$ and $\sum_{j=1}^{n} b_j Y_j$?

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Correlation

- We know that the sign of a covariance indicates whether $X - E[X]$ and $Y - E[Y]$ tend to have the same sign.
- The magnitude gives us some indication of the extent to which this is true... but it is hard to interpret.
  - The magnitude depends not just how much X and Y co-vary, but also on how much X and Y deviate from their expected values.
- The **correlation coefficient** $\rho_{X,Y}$(sometimes referred to as the Pearson's correlation coefficient) is a standardized version of the covariance.

$$\rho_{X,Y} = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}$$

- We always have $-1 \leq \rho_{X,Y} \leq 1$
  - $\rho = 0$ implies zero covariance.
  - $|\rho| = 1$ iff there is a linear relationship between X and Y.

Correlation: Example of $|\rho| = 1$

- We throw a biased coin, with probability of heads p, n times. Let X be the number of heads, and let Y be the number of tails.
- $Y = n - X$
- $E[X] =$

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- Remember $X = n - Y$, so they have a linear relationship.

Correlation: Example of $|\rho| = 1$

- Let $Y = aX + b$.

Correlation: Example of $|\rho| = 1$

- Let $Y = aX + b$.
- $$\text{cov}(X, aX + b) = \text{cov}(X, aX) = a\text{var}(X)$$
$$\rho_{X,aX+b} = \frac{\text{cov}(X, aX)}{\sqrt{\text{var}(X)a^2\text{var}(X)}} = \frac{a\text{var}(X)}{|a|\text{var}(X)} = \frac{a}{|a|}$$
$$= \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases}$$

Variance of a sum of random variables

- Earlier in the course, we looked at the variance of the sum of discrete random variables. Same rule holds for continuous random variables.

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)$$

- So for independent X, Y, $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$

More than two random variables

- For multiple random variables we have:
  $P((X,Y,Z) \in B) = \int_{(x,y,z)\in B)} f_{X,Y,Z}(x,y,z)dxdydz$
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- Independence: $f_{X,Y,Z}(x, y, z) = f_X(x) f_Y(y) f_Z(z)$ For all x, y, z

More than two random variables

- For two random variables X, Y arising out of the same experiment, we define their CDF as:

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y) =$$

- How do I get $f_{X,Y}(x, y)$ back? $f_{X,Y}(x, y) = \dfrac{d^2 F_{X,Y}(x, y)}{dx\ dy}$
- Let X and Y be jointly uniform on the unit square. $F_{X,Y}(x, y) = xy$ for $0 \leq x, y \leq 1$
- What is $f_{X,Y}(x, y)$?. Differentiate! $\dfrac{d}{dx}\left(\dfrac{d}{dy}(xy)\right)$
- This equals 1 for all $0 \leq x, y \leq 1$!

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- $X \sim Exp(1)$. Find the PDF of $Y = X^2$.
- How do we do this?
- Start with the CDF. Find $F_Y(y) = P(X^2 \leq y)$.

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- So $f_Y(y) = \begin{cases} \frac{dF_Y(y)}{dy}x = \frac{\exp(-\sqrt{y})}{2\sqrt{y}} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Function of random variables-II

- $X \sim Uniform([0,1])$. Find the PDF of $Y = X^2$.
- How do we do this?
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Function of random variables-II

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