THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 18: Continuous random variables: conditional expectation, Independence, covariance

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Roadmap

- Two random variables: joint distributions
  - Joint pdf
  - Joint pdf to a single pdf: Marginalization
  - Conditional pdf
    - Conditioning on an event
    - Conditioning on a continuous r.v
    - Total probability rule for continuous r.v's
    - Bayes theorem for continuous r.v's
    - Conditional expectation and total expectation theorem
  - Independence
  - Covariance and correlation.
- More than two random variables.

Bayes' law with continuous outcomes but discrete hidden causes

- Sometimes our hidden cause is inherently discrete.
  - e.g. I may be interested in whether I have flu or not – a binary choice.
  - My observation might be my temperature – a continuous random variable.
- We want $P(A|Y = y) =$ e.g. $P(\text{flu}|Y = 100)$
- Pretend Y is a discrete r.v.

$$P(A|Y = y) = \frac{P(Y = y|A)P(A)}{P(Y = y|A)P(A) + P(Y = y|A^c)P(A^c)}$$

All that changes for a continuous r.v. is:

$$P(A|Y = y) = \frac{f_{Y|A}(y)P(A)}{f_{Y|A}(y)P(A) + f_{Y|A^c}(y)P(A^c)}$$

Bayes' law with continuous outcomes but discrete hidden causes

- The probability that anyone has flu (event A) is 20%.
- Body temperature is Y.
- Without flu, Y is a normal random variable with $\mu = 98.6$ degrees and $\sigma = .5$.
- With flu, Y is a normal random variable with $\mu = 102$ and $\sigma = 2$.
- My temperature is 100. If A is the event "has flu" and Y is temp.

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- My temperature is 100. If A is the event "has flu" and Y is temp.
- $$\boxed{\begin{aligned} f_{Y|A}(y) &= \frac{1}{\sqrt{2\pi \times 4}} \exp - \frac{(y-102)^2}{2 \times 4} \\ f_{Y|A^c}(y) &= \frac{1}{\sqrt{2\pi \times .25}} \exp - \frac{(y-98.6)^2}{2 \times .25} \end{aligned}}$$

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- With flu, Y is a normal random variable with $\mu = 102$ and $\sigma = 2$.
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- $$f_{Y|A}(y) = \frac{1}{\sqrt{2\pi \times 4}} \exp - \frac{(y-102)^2}{2 \times 4}$$
  $$f_{Y|A^c}(y) = \frac{1}{\sqrt{2\pi \times .25}} \exp - \frac{(y-98.6)^2}{2 \times .25}$$
- $$P(A|Y=y) = \frac{P(A) f_{Y|A}(y)}{f_Y(y)} = \frac{f_{Y|A}(y)P(A)}{f_{Y|A}(y)P(A) + f_{Y|A^c}(y)P(A^c)}$$
  $$P(A|Y=100) = \frac{0.2 \frac{1}{2\sqrt{2\pi}} e^{-(100-102)^2/8}}{0.2 \frac{1}{2\sqrt{2\pi}} e^{-(100-102)^2/8} + 0.8 \frac{1}{0.5\sqrt{2\pi}} e^{-(100-98.6)^2/0.5}} = 0.65$$

Conditional Expectation

- When we were looking at discrete random variables, we looked at **conditional expectations**.
- The conditional expectation, $E[X|A]$, of a random variable X given an event A is the value of X we expect to get out, on average, when A is true.
- We could calculate it by summing over all values x that X can take on, and scaling them by the conditional PMF $p_{X|A}(x) = P(X = x|A)$.

$$E[X|A] = \sum_x x p_{X|A}(x)$$

Conditional Expectation

- We can also look at the conditional expectation of a continuous random variable.
- If $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$, what do you think the conditional expectation of X given some event A looks like?

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- How about the conditional expectation of some function X given another random variable Y?
- How about the conditional expectation of some function $g(X)$ given some event A?
- $E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x) dx$

Total expectation theorem

- More generally, if $A_1, A_2, \ldots, A_n$ are a partition of Ω, we have a continuous version of the **total expectation theorem**:

$$E[X] = \sum_{i=1}^{n} P(A_i) E[X|A_i]$$

Conditional expectation

- I am expecting an email, that will definitely arrive between midday and 3pm.
- Within a given hour (midday-1, 1-2, 2-3), each time is equally likely.
- It is twice as likely to arrive between 1 and 2 as it is to arrive between midday and 1.
- It is twice as likely to arrive between 2 and 3 as it is to arrive between 1 and 2.
- What does the PDF look like?

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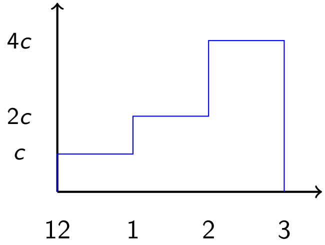

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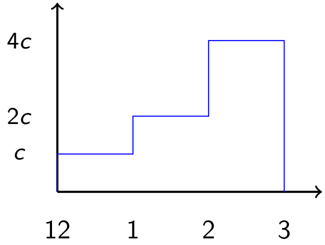

- What is c?

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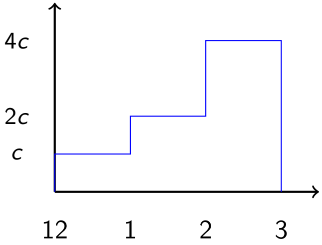
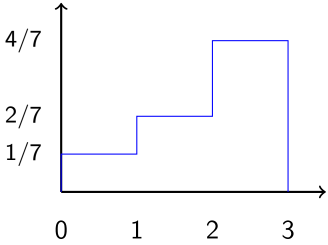


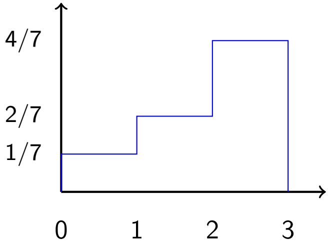
- What is c? $1/7$

Conditional expectation



- I wait until 2pm. It still hasn't arrived. What is the expected value of the arrival time?
- What is the expected time without any conditioning?

Conditional expectation



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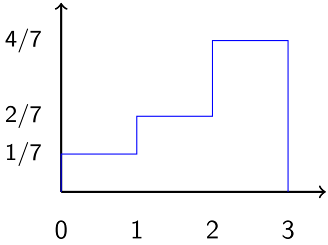

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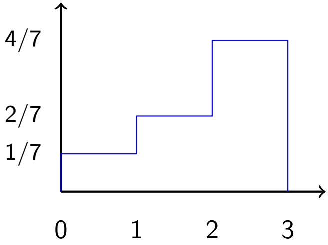

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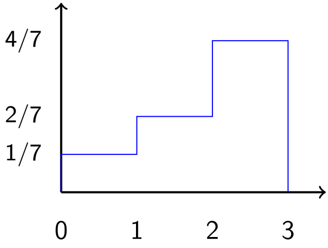
$$f_{X|X>2}(x) = \begin{cases} 1 & \text{if } 2 < x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Conditional expectation



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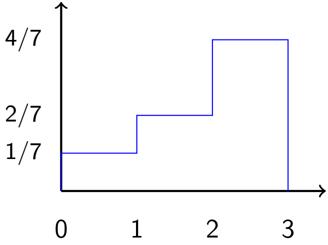
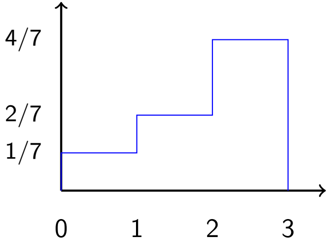
- So, $E[X|X > 2] = \int_{-\infty}^{\infty} x f_{X|X>2}(x) dx = \int_2^3 x\, dx = 2.5.$

Conditional expectation



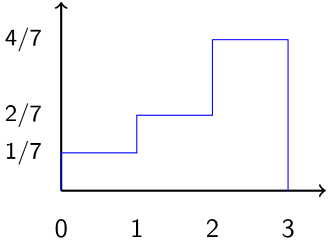
- What is the (unconditional) probability that $X > 2$?

Conditional expectation



- What is the (unconditional) probability that $X > 2$?
- $P(X > 2) = \int_2^3 f_X(x)dx = 4/7$

Conditional expectation



- What is the (unconditional) probability that $X > 2$?
- $P(X > 2) = \int_2^3 f_X(x)dx = 4/7$
- Similarly, $P(X < 1) = \int_0^1 f_X(x)dx = 1/7$ and $P(1 \leq X \leq 2) = 2/7$.

Total expectation theorem

- What is the total expectation of X?

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- $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$
- By the total probability theorem,

$$\begin{aligned} f_X(x) = &P(X \leq 1) f_{X|0 \leq X \leq 1}(x) \\ &+ P(1 \leq X \leq 2) f_{X|1 \leq X \leq 2}(x) + P(X > 2) f_{X|X>2}(x) \end{aligned}$$

Total expectation theorem

- What is the total expectation of X?
- $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$
- By the total probability theorem,

$$
\begin{aligned}
f_X(x) =& P(X \leq 1) f_{X|0 \leq X \leq 1}(x) \\
&+ P(1 \leq X \leq 2) f_{X|1 \leq X \leq 2}(x) + P(X > 2) f_{X|X>2}(x)
\end{aligned}
$$

- So, we can write the total expectation as

$$
\begin{aligned}
E[X] = & \int_0^1 x P(X \leq 1) f_{X|X \leq 1}(x) + \int_1^2 x P(1 \leq X \leq 2) f_{X|1 \leq X \leq 2}(x) \\
& + \int_2^3 x P(X > 2) f_{X|X>2}(x) \\
=& E[X|X \leq 1] P(X \leq 1) + E[X|1 \leq X \leq 2] P(1 \leq X \leq 2) \\
& + E[X|X > 2] P(X > 2) \\
=& 0.5 \cdot 1/7 + 1.5 \cdot 2/7 + 2.5 \cdot 4/7 = 27/14
\end{aligned}
$$

Independent random variables

- For discrete random variables, we said two random variables X and Y are independent if

$$p_{X,Y}(x,y) = p_X(x)p_Y(y) \qquad \forall x, y$$

- Just like in the discrete case, we say two continous random variables are independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \qquad \forall x, y$$

- If $f_Y(y) > 0$, this is the same as saying $f_X(x) = f_{X|Y}(x|y)$ – i.e. knowing that $Y = y$ doesn't tell us anything about X.
- Just like with discrete random variables, we if X and Y are independent we have $E[XY] = E[X]E[Y]$ and $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$.
  - For two functions $f(X)$ and $g(Y)$ we have $E[f(X)g(Y)] = E[f(X)]E[g(Y)]$.

Independent random variables-example

- You have two random variables X, Y with joint PDF

$$f_{XY}(x, y) = \begin{cases} cxy & x, y \in [0, 1] \\ 0 & \text{Otherwise} \end{cases}$$

- What is c?
- Are X, Y independent?

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- $c \int_{x=0}^{1} \int_{y=0}^{1} xydydx = c/4 = 1$. So $c = 4$.

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- $c \int_{x=0}^{1} \int_{y=0}^{1} xydydx = c/4 = 1$. So $c = 4$.
- $$f_X(x) = \begin{cases} \int_0^1 4xydy = 2x & x \in [0,1] \\ 0 & \text{Otherwise} \end{cases}$$

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- $c \int_{x=0}^{1} \int_{y=0}^{1} xydydx = c/4 = 1$. So $c = 4$.
- $f_X(x) = \begin{cases} \int_0^1 4xydy = 2x & x \in [0,1] \\ 0 & \text{Otherwise} \end{cases}$
- $f_Y(y) = \begin{cases} 2y & y \in [0,1] \\ 0 & \text{Otherwise} \end{cases}$
- $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for all $x, y \in [0,1]$.