Homework Assignment 9

Due on Wednesday April 28th by midnight via Canvas

SDS 321 Intro to Probability and Statistics

1 Questions

1. (3+1+3+1 pts) If X, Y are independent and identically distributed random variables having uniform distributions over $[0,1]$. Let $Z = \max(2X, Y)$ and $U = \min(X, Y)$.

   (a) Find $f_Z(z)$

$$\begin{aligned} F_Z(z) &= P(\max(2X,Y) \leq z) = P(2X \leq z, Y \leq z) = P(X \leq z/2)P(Y \leq z) \\ &= \begin{cases} z/2 \times z = z^2/2 & z \in [0,1] \\ z/2 \times 1 = z/2 & z \in [1,2] \end{cases} \end{aligned}$$

$$f_Z(z) = \begin{cases} z & z \in [0,1] \\ 1/2 & z \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

   If they just do one part, then give them half the points and point to the solutions.

   (b) Find $E[Z]$

$$\begin{aligned} E[Z] &= \int_0^2 z f_Z(z)dz = \int_0^1 z^2 dz + \int_1^2 z/2 dz \\ &= 1/3 + 1/4(2^2 - 1) = 1/3 + 3/4 = 13/12 \end{aligned}$$

   (c) Find $f_U(u)$ For $z \in [0,1]$

$$\begin{aligned} F_Z(z) &= P(\min(X,Y) \leq z) = 1 - P(\min(X,Y) \geq z) \\ &= 1 - P(X \geq z, Y \geq z) = 1 - (1-z)^2 \end{aligned}$$

$$f_Z(z) = \frac{d}{dz} 1 - (1-z)^2 = 2(1-z)$$

   (d) Find $E[U]$ $E[U] = \int_0^1 2z(1-z)dz = 2(1/2 - 1/3) = 1/3.$

2. (4 pts) Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability .6, compute the expected number of ducks that are hit, given that the number of

ducks flying at that time is k. By using the fact that a hunter can only hit one duck, $P(\text{Hunter i hits some duck}) = \sum_j P(\text{Hunter i hits duck j}) = kP(\text{Hunter i hits duck 1}) = .6$. So $P(\text{Hunter i hits duck j}) = .6/k$ for $i = 1, \dots 10, j = 1, \dots k$. Now if $X_i = 1$ if i^{th} duck is hit, then we want $E[X_1 + \cdots + X_k] = kE[X_1] = kP(\text{Duck 1 is hit})$. But the probability that duck 1 is hit is the same as some hunter has hit it, which is one minus the probability that no hunter has hit it. $P(\text{Duck 1 is hit}) = 1 - P(\cap_{i=1}^{1} 0\{\text{Hunter i did not hit duck 1}\}) = 1 - (1 - .6/k)^{10}$

Solution to the original question which asks for expected number of ducks hit, if the number of ducks flying in a flock was a Poisson(6) random variable.

If the question was originally put, then you will have something more complicated. Consider now that the number of ducks is not k, but a Poisson random variable $K \sim Poisson(6)$. Let Y be the number of ducks hit. You have calculated $E[Y|K = k]$. So now you answer will be:

$$E[Y] = E[E[Y|K]] = \sum_{k=0}^{\infty} E[Y|K = k]P(K = k) = \sum_{k=0}^{\infty} k \left(1 - (1 - .6/k)^{10}\right) \frac{e^{-6}6^k}{k!}$$

3. (2+2+2+2 = 8pts) From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75.

   (a) Give an upper bound for the probability that a student's test score will exceed 85. Use Markov's inequality. $P(X > 85) \leq E[X]/85 = 75/85 = 15/17$

   (b) Suppose, in addition, that the professor knows that the variance of a student's test score is equal to 25.

      i. What can be said about the probability that a student will score between 65 and 85? Here we use Chebyshev. $P(65 \leq X \leq 85) = P(|X - 75| \leq 10) = 1 - P(|X - 75| > 10) \geq 1 - \text{var}(X)/10^2 = 1 - 25/100 = 3/4$

      ii. How many students would have to take the examination to ensure, with probability at least .9, that the class average would be within 5 of 75? Do not use the central limit theorem. $P(|\bar{X} - 75| \leq 5) = 1 - P(|\bar{X} - 75| > 5) \geq 1 - \text{var}(\bar{X})/25 = 0.9$. So $1/n = .1$ and so $n = 10$.

      iii. Now calculate the last part with the central limit theorem. $P(|\bar{X} - 75| \leq 5) = P(|Z| \leq \sqrt{n}) = 0.9$ and so $\Phi(\sqrt{n}) = .95$. A normal table lookup gives $\sqrt{n} = 1.65$ and so $n \approx 3$. This is a situation where you should not trust the answer, since for $n = 3$ a CLT approximation is not valid anyway. However the Chebyshev is valid and gives a correct but possibly not tight answer.