Homework Assignment 7

Due via Canvas, **Friday** March 31st

SDS 321 Intro to Probability and Statistics

1. (2+2) Let X be a random variable with PDF

$$f(x) = \begin{cases} C(1-x^2) & -1 < x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

   (a) What is C? By normalization $\int_{-1}^{1} C(1-x^2)dx = C\left(x - x^3/3\right)\Big|_{-1}^{1} = 1$. So $C(2 - 2/3) = 4C/3 = 1$. So $C = 3/4$.

   (b) What is the CDF (Cumulative Distribution Function) of X?
   $F_X(t) = \int_{-\infty}^{t} f_X(x)dx$. So we have:

$$F_X(t) = \begin{cases} 0 & t < -1 \\ 3/4(t - t^3/3 + 2/3) & -1 \leq t \leq 1 \\ 1 & t > 1 \end{cases}$$

2. (2+2+2) The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & \text{Otherwise} \end{cases}$$

   (a) $P(X > 15) =?$
   $P(X > 15) = 1 - \int_{10}^{15} \frac{10}{x^2}dx = 1 - (-10/x)|_{10}^{15} = 1 - (-10/15 + 1) = 2/3.$

   (b) What is the CDF of X?

$$F_X(t) = \begin{cases} 0 & t < 10 \\ 1 - 10/t & 10 \leq t \end{cases}$$

   (c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?
   Let X be the number of devices will function at least 15 hours. $X \sim Binomial(6, 2/3)$. So $P(X \geq 3) = \sum_{i=3}^{6} \binom{6}{i}(2/3)^i(1/3)^{6-i} = .9$.

3. (3+2) The PDF of X is given by:

$$f_X = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) If $E[X] = 3/5$, what are a and b? Normalization gives $a + b/3 = 1$. Expectation gives $a/2 + b/4 = 3/5$. Solving we get: $b = 6/5$ and $a = 3/5$.

(b) What is $\text{var}(X)$? $E[X^2] = \int_0^1 (3/5+6/5x^2)x^2 dx = 1/5+6/25 = 11/25$. So $\text{var}(X) = 11/25 - (3/5)^2 = 2/25$.

4. (3 + 2 pts) Suppose that X is a normal random variable with mean 5. If $P(X > 9) = .2$.

   (a) Approximately what is $\text{var}(X)$? $P((X-5)/\sigma > (9-5)/\sigma >) = \phi(-4/\sigma) = .2$. $4/\sigma \approx .8514$, so $\text{var}(X) \approx 22.59$.

   (b) With this variance, calculate $P(|X - 5| > 4)$. $P(|X-5| > 4) = P(X > 9) + P(X < 1) = 2 \times .2 = .4$.

Extra credit (10 pts)

5. (2+2 pts) You arrive at a bus stop at 10 oclock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

   (a) What is the probability that you will have to wait longer than 10 minutes?
   Let X denote the time after which bus arrives. $X \sim Uniform([0, 30])$. $P(X > 10) = 20/30 = 2/3$.

   (b) If, at 10:10, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?
   Let X denote the time after which bus arrives. $P(X > 20|X > 10) = P(X > 20)/P(X > 10) = 1/2$.

6. (1+5 pts) A point is chosen uniformly at random on a line segment of length L.

   (a) What is the PDF of the position of the point. Take the left endpoint of the line as the origin.
   $X \sim Uniform(0, L)$.

$$f_X(x) = \begin{cases} 1/L & 0 \le x \le L \\ 0 & \text{Otherwise} \end{cases}$$

   (b) Find the probability that the ratio of the shorter to the longer segment is less than 1/4.
   If $x < L/2$ then smaller to larger ratio would be: $x/(L-x) < 1/4$, i.e. $x < L/5$. If $x > L/2$ then the ratio would be: $(L-x)/x < 1/4$, i.e. $x > 4L/5$. So $P(X < L/5 \cup X > 4L/5) = 1 - 3/5 = 2/5$.

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990