Homework Assignment 5

Due Friday Feb 24th by 5pm via Canvas

SDS 321 Intro to Probability and Statistics

1. (2+2+2) $X \sim Bernoulli(p)$. Define a random variable $Y := 2^X 3^{1-X}$.

   (a) Write down the PMF of Y. When $X = 1$, $Y = 2$ and when $X = 0$, $Y = 3$. So Y takes two values.

   $$P(Y = k) = \begin{cases} p & \text{When } k = 2 \\ 1 - p & \text{When } k = 3 \end{cases}$$

   (b) What is $E[Y]$? 2p+3(1-p)=3-p

   (c) What is var(Y)? Note that Y=3-X. So var(Y) = var$(X) = p(1 - p)$

2. (1+4+1) 3 German Shepherds and 3 Siberian Huskies enter a dog show. At the end of the show a ranking of the 6 dogs is provided. Assume that all rankings are equally likely. Let X denote the highest ranking obtained by a German Shepherd. For example, $X = 1$ if a German Shepherd is the show winner.

   (a) What values can X take? 1,2,3,4

   (b) Write down the PMF. 3 ways to put a GS in the first place.

   $$P(X = 1) = P(\text{A GS is in the first place. }) = \frac{3 \times 5!}{6!} = 1/2$$

   $$P(X = 2) = P(\text{A SH is in the first place and a GS is in the second place}) = \frac{3 \times 3 \times 4!}{6!} = 3/10$$

   $$P(X = 3) = P(\text{All SH are in the first 2 places and a GS in the } 3^{rd} \text{ place}) = \frac{3 \times 2 \times 3 \times 3!}{6!}$$

   $$P(X = 4) = P(\text{All SH are in the first 3 places and a GS in the } 4^{rd} \text{ place}) = \frac{3! \times 3!}{6!}$$

   (c) What is $E[X]$? $1 \times 1/2 + 2 \times 3/10 + 3 \times 3/20 + 4 \times 1/20$.

3. (2+2+4)You have a fair coin and a biased coin. The first biased coin lands heads with probability .5 whereas the second lands head with probability .8. You choose one of these coins at random and now toss it 10 times.

(a) What is the probability that the coin will land on heads exactly 5 out of 10 times?
Let $F = \{\text{You picked a fair coin}\}$ $P(X = 5) = P(X = 5|F)P(F)+P(X = 5|F^c)P(F^c) = \binom{10}{5}/2^{10} + \binom{10}{5}.8^5.2^5$

(b) What is the probability that the coin will land on a head in the first toss? Let $A = \{\text{Head in first toss}\}$ $P(A) = P(A|F)P(F) + P(A|F^c)P(F^c) = .5 \times .5 + .8 \times .5 = .65.$

(c) What is the probability that the coin will land on heads exactly 5 out of 10 times given that it lands on a head in the first toss? $P(X = 5|A, F) = P(\{\text{last 9 tosses has 4 heads—F}\})$

$$\begin{aligned} P(X = 5|A) &= \frac{P(X = 5, A)}{P(A)} \\ &= \frac{P(X = 5, A|F)P(F) + P(X = 5, A|F^c)P(F^c)}{P(A)} \\ &= \frac{P(A|F)P(X = 5|A, F)P(F) + P(A|F^c)P(X = 5|A, F^c)P(F^c)}{.65} \\ &= \frac{.5\binom{9}{4}/2^9 \times .5 + .8 \times \binom{9}{4}.8^4.2^5 \times .5}{.65} \end{aligned}$$