Homework Assignment 10

Due on Sunday May 7th by midnight via Canvas

SDS 321 Intro to Probability and Statistics

1. (3+3+3+1 pts) Let $X_1, \ldots, X_n$ be i.i.d draws from $Uniform([0, a])$.

   (a) Find the MLE of a.

   (b) Find the pdf of the MLE. *Hint: you can use your knowledge on how to get the pdf of the max of n i.i.d r.v's.*

   (c) What is the expectation of the MLE?

   (d) Is this unbiased, or asymptotically unbiased?

   Write the likelihood
   $$p_X(X; a) = \frac{\mathbf{1}(0 \leq X_1, \ldots, X_n \leq a)}{a^n}$$
   Clearly $\max(X_1, \ldots, X_n) \leq a$, and the a that maximizes the above is $\hat{a} = \max(X_1, \ldots, X_n)$. Now we need to get the distribution of this in order to calculate the expectation.
   $$F_{\hat{a}}(t; a) = P(X_1, \ldots, X_n \leq t; a) = (F_{X_1}(X_1 \leq t; a))^n = (t/a)^n$$
   $$f_{\hat{a}}(t; a) = \frac{nt^{n-1}}{a^n}$$
   So the expectation is given by:
   $$E[\hat{a}] = \int_0^a t\frac{nt^{n-1}}{a^n}dt = \int_0^a n\frac{t^n}{a^n}dt = \frac{n}{n+1}a$$
   Clearly, this is not unbiased, but asymptotically unbiased. One point for writing the joint pdf properly. One point for the estimator. One point for correct answer asymptotically unbiased. 2 points for calculating the CDF of the MLE. One point for differentiating and getting the pdf. 2 points for calculating the correct expectation (1 pt for setting up the integral with limits and 1 pt for evaluating it right).

2. (2+3 pts) Let $X_1, \ldots, X_n$ be i.i.d draws from $Normal(\mu, \sigma^2)$.

   (a) Show that the MLE for σ^2 is $\sum_i (X_i - \bar{X})^2/n$.

   (b) Use the fact that $\sum_i (X_i - \bar{X})^2/n = \sum_i X_i^2/n - \bar{X}^2$ to prove that the MLE you have from above is asymptotically unbiased. *Hint: We didn't get time to do this in class. But this is on the slides for 27th April, lecture 24.*

3. (1+2+2 pts) You have collected the average annual precipitation of Austin from the last 6 years. These are 31.8621 32.6414 32.4491 35.3799 28.5916 35.5605. You can assume that these are independent draws from a normal distribution with unknown mean μ and variance σ^2. You are trying to test the hypothesis $H_0 : \mu = 35$ and the alternative is $H_1 : \mu \neq 35$.

(a) Estimate μ and σ^2 from the above. You can use the MLE of σ^2 to estimate the variance. Sample mean 32.7474 and sample variance 6.6. Note, if you used MLE it is 5.5, but the unbiased estimator gives 6.6.

(b) An oracle told you that $\sigma = 5$. Test the null hypothesis at 5% significance level. A fair choice seems like we should reject when $|\bar{X} - 35| > \xi$. Under H_0, $\bar{X} \sim N(35, 2.04)$ $P(|\bar{X} - \mu| > \xi/1.04; H_0) = P(|Z| > \xi/2.04) = 0.05$ and so $\xi/2.04 = \Phi^{-1}(.975)$ and $\xi = 2.04 \times 1.96 \approx 4$. So your rejection region is $\{(x_1, \ldots, x_n) : |\bar{x} - 35| > 4\}$. Now your calculated $\bar{x}$ is 32.74 which is not in this region. So accept.

(c) You don't know the true σ. Now test the null hypothesis at 5% significance level. I don't know σ, but I can estimate it using the unbiased estimator of the sq. root of the sample variance. But under H_0, $(\bar{X} - \mu)/(S/\sqrt{6})$ is from the t distribution with 5 degrees of freedom. So we want $P(|T_5| > \xi/(1.05)) = 0.05$. So $\xi/1.05 = 2.57$. As a result, the rejection region is $R = \{x||\bar{X} - 35| > 2.67\}$. Now from the calculated data, $|\bar{x} - 35| = 2.25$ which is still barely in the acceptance region. So accept.

Standard normal table

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

t-table

	0.100	0.050	0.025	0.010	0.005	0.001
1	3.078	6.314	12.71	31.82	63.66	318.3
2	1.886	2.920	4.303	6.965	9.925	22.33
3	1.638	2.353	3.182	4.541	5.841	10.21
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
11	1.363	1.796	2.201	2.718	3.106	4.025
12	1.356	1.782	2.179	2.681	3.055	3.930
13	1.350	1.771	2.160	2.650	3.012	3.852
14	1.345	1.761	2.145	2.624	2.977	3.787
15	1.341	1.753	2.131	2.602	2.947	3.733
20	1.325	1.725	2.086	2.528	2.845	3.552
30	1.310	1.697	2.042	2.457	2.750	3.385
60	1.296	1.671	2.000	2.390	2.660	3.232
120	1.289	1.658	1.980	2.358	2.617	3.160
∞	1.282	1.645	1.960	2.326	2.576	3.090