Homework 2

CMU 10-708: Probabilistic Graphical Models (Fall 2020)

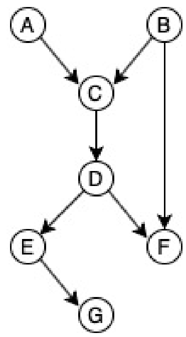
October 4, 2020

Instructions:

- **Collaboration policy:** Collaboration on solving the homework is allowed, after you have thought about the problems on your own. It is also OK to get clarification (but not solutions) from books or online resources, again after you have thought about the problems on your own. There are two requirements: first, cite your collaborators fully and completely (e.g., "Bob explained to me what is asked in Question 4.3"). Second, write your solution *independently*: close the book and all of your notes, and send collaborators out of the room, so that the solution comes from you only.
- **Submitting your work:** Assignments should be submitted as PDFs using Gradescope unless explicitly stated otherwise. Each derivation/proof should be completed on a separate page. Submissions can be handwritten, but should be labeled and clearly legible. Else, submission can be written in LaTeX. Upon submission, label each question using the tempate provided by Gradescope.

1 DGMs to UGMs and back (Chang)[20 pts]

1. Answer the following questions based on the graph.



   (a) **[1 points]** $P(A|B, D, F) \stackrel{?}{=} P(A|D, F)$

   (b) **[1 points]** $P(A|B) \stackrel{?}{=} P(A)$

   (c) **[1 points]** $P(C|B, F) \stackrel{?}{=} P(C|B)$

   (d) **[1 points]** $P(E|D, F) \stackrel{?}{=} P(E|D)$

   (e) **[2 points]** $P(D|B, E, F, G) \stackrel{?}{=} P(D|E, F)$

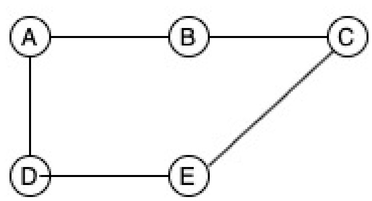
2. **[10 points]** Prove that for any chordal undirected graph $\mathcal{G}$, any minimal separator between two non-adjacent vertices x and y on the graph is a clique.

   Hints:

   (a) Consider three vertex sets: a separator S, a clique X that contains x and a clique Y that contains y.

   (b) Prove by contradiction. If minimal S is not a clique, there exists a non-chordal loop of length ≥ 4. And so on.

3. Consider the following directed graph $\mathcal{G}$:

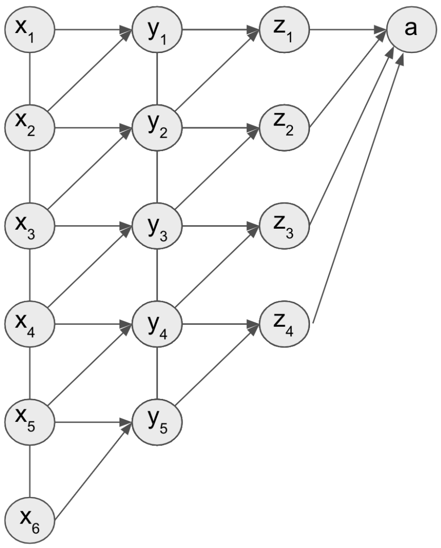


   (a) **[1 points]** Triangulate $\mathcal{G}$ to construct a chordal graph $\mathcal{H}$.

   (b) **[3 points]** Construct a minimal DAG with the same conditional independencies as $\mathcal{H}$.

2 Chain Graphical Models (Chih-Kuan) [25 pts]

Consider the chain graphical model in the figure. Here, $A \in \{0, 1\}$ is binary variable, while $X_i, Y_i, Z_i \in \mathbb{R}$ are all real numbers. We denote $X = \{X_1, X_2, X_3, X_4, X_5, X_6\}$, $Y = \{Y_1, Y_2, Y_3, Y_4, Y_5\}$, $Z = \{Z_1, Z_2, Z_3, Z_4\}$. (We use (A, a), (X, x), (Y, y), (Z, z) interchangeably in this question.) We are given that $P(a|z) \propto \exp\left(\sum_{i=1}^{4} \phi_i(z_i, a)\right)$, where $\phi_i(z_i, a) = w_i z_i a$.



1. **[4 points]** Write down the factorization of the joint distribution $P(A, X, Y, Z)$ with respect to the chain graph.

2. **[10 points]** Suppose $X_i \in \mathbb{N}$, and consider the sequence of integers $(x_1, x_2, x_3, x_4, x_5, x_6)$. Set the factors of the joint distribution, and weights w_1, w_2, w_3, w_4 specifying $P(A|Z)$, so that $P(A = 1|x) > 0.5$ if $(x_1, x_2, x_3, x_4, x_5, x_6)$ contains two identical numbers at a distance of 2 (For example, $(1, 1, 2, 2, 3, 3)$ does not contain identical numbers at a distance of 2, but $(1, 2, 3, 2, 5, 6)$ has identical numbers at a distance of 2), and $P(A = 1|x) < 0.5$ otherwise.

   Hint: Use indicator functions.

3. **[3 points]** Draw the moralization of the chain graph.

4. **[4 points]** (c-separation) Answer whether the following pairs are c-separated (give explanation when it is not separated):

   (a) $\text{CSEP}(\{X_1, X_2\}, \{Z_1, Z_2\}|\{Y_1, Y_2\})$

   (b) $\text{CSEP}(\{X_1, X_2\}, \{Z_1, Z_2\}|\{Y_1, Y_2, Y_3\})$

   (c) $\text{CSEP}(\{X_1, X_2, X_3\}, \{A\}|\{Y_1, Y_2, Y_3, Y_4, X_5\})$

   (d) $\text{CSEP}(X_1, Y_2|X_2)$

5. **[4 points]** (AMP-separation) Answer whether the following pairs are AMP-separated (give explanation when it is not separated):

   (a) $\text{AMP}(\{X_1, X_2\}, \{Z_1, Z_2\}|\{Y_1, Y_2\})$

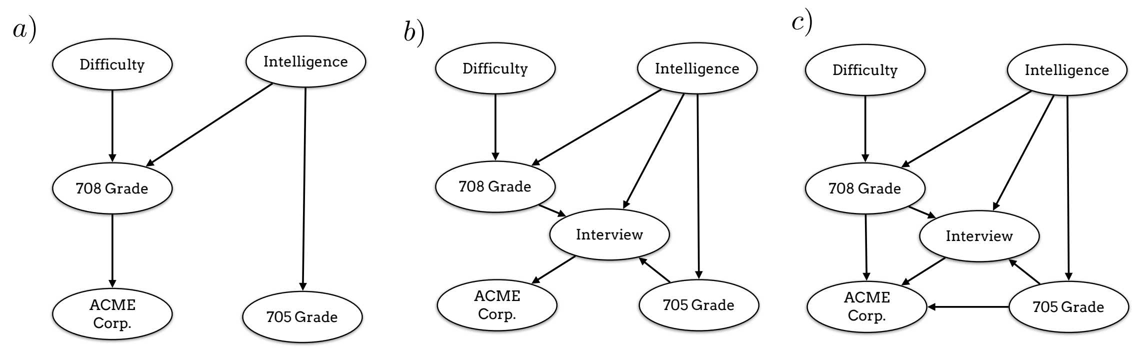
   (b) $\text{AMP}(\{X_1, X_2\}, \{Z_1, Z_2\}|\{Y_1, Y_2, Y_3\})$

   (c) $\text{AMP}(\{X_1, X_2, X_3\}, \{A\}|\{Y_1, Y_2, Y_3, Y_4, X_5\})$

   (d) $\text{AMP}(X_1, Y_2 \mid X_2)$

3 Structural Causal Models, Compliance (Chirag) [23 pts]

1. Consider the figures below to answer the following questions.



   (a) **[1 points]** Forrest Gump is a graduate student at CMU and wants to get a job at ACME Corp. Gump has heard that ACME only hires students who have received an A in 10-708. Gump's belief of the Causal model is given in Figure A above. Unfortunately Gump scored a B+ on 708 and so decides to hack into CMU SIO and alter his grade to A.

   - Describe how the Causal Graph would change after Gump has performed this intervention.
   - Would the final decision of him being hired at ACME Corp. be independent of his intelligence post the intervention ?

   (b) **[1 points]** ACME Corp decides to instead change their hiring process and include an interview (Figure B). Gump still goes ahead and hacks into SIO, alters his 708 Grade and goes ahead with the interview. The interviewer asks Gump of his 708 and 705 score and the interview goes well.

   - Describe how the Causal Graph would change after Gump has performed the intervention and completed the interview.
   - Would the final decision of him being hired at ACME Corp. now be independent of his intelligence post the intervention ?

   (c) **[1 points]** The independent hiring committee at ACME Corp makes the hiring process more stringent and decided to reconsider the 708 and 705 grades along with the interviewers feedback to make the final decision (Fig. C). As originally planned Gump still alters the 708 grade and still believes the interview went well.

   - Describe how the Causal Graph would change after Gump has performed the intervention and completed the interview.
   - Would the final decision of him being hired at ACME Corp. now be independent of his intelligence post the intervention ?

2. In the following questions we describe a hypothetical scenario and ask you to describe it using an appropriate structural causal model.

   (a) **[10 points]** The Queen of Lilliput claims that scientists from Lilliput have developed new vaccine for SARS-CoV-2 and confirmed its efficacy with a large randomized control trial. Consider a causal model representing the randomized control trial, where $O \in \{0, 1\}$

represents a binary outcome and $A \in \{0, 1\}$ represents a binary assignment to a treatment or a placebo. The assignment was carried out at random as $A \sim \text{Bernoulli}(0.5)$.

Describe this scenario with a structural causal model. Specify the exogenous and endogenous variables and the interventional distributions. (**Hint**: Since we assume that the variables are Binary, notice that there would only be 4 possible ways in which treatment assignment may effect the outcome.)

(b) **[10 points]** Unfortunately though, some Lilliputians are known to be defiant and not obey their Queen. A large number of Lilliputians are believed to have not taken the vaccine as prescribed and there are also media reports that some obtained the true vaccine illegally and self administered the vaccine. Consider the following classes of Lilliputians:

- Frightened Lilliputians take the vaccine whether assigned or not.
- Disciplined Lilliputians takes the vaccine if and only if prescribed.
- Defiant; take the vaccine only if not prescribed.
- Apathetic; Never take the vaccine.

Assume that we have access to the distribution, C that determines the class of Lilliputians.

Describe this alternate scenario with a structural causal model, describe the set of endogenous and exogenous variables and functional responses describing the interventional queries.