Independent Component Analysis (ICA)

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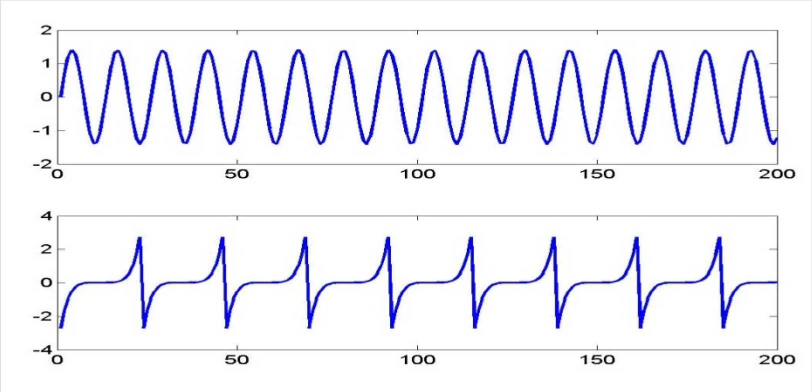
Machine Learning 10-701

Slides courtesy of Barnabas Poczos

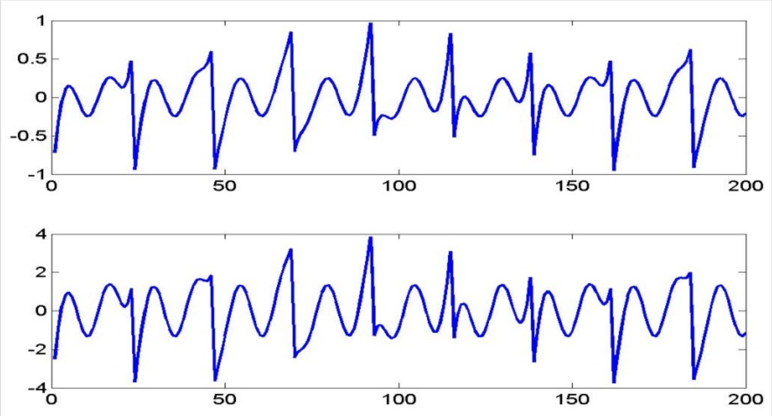
Independent Component Analysis

$$
\begin{aligned}
x_1(t) &= a_{11}s_1(t) + a_{12}s_2(t) \\
x_2(t) &= a_{21}s_1(t) + a_{22}s_2(t)
\end{aligned}
$$

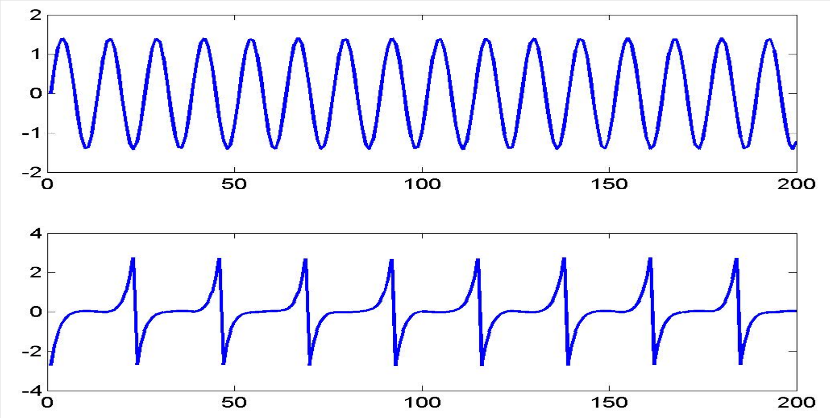
Model



original signals



Observations (Mixtures)



ICA estimated signals

Independent Component Analysys

Model

$$\begin{aligned} x_1(t) &= a_{11}s_1(t) + a_{12}s_2(t) \\ x_2(t) &= a_{21}s_1(t) + a_{22}s_2(t) \end{aligned}$$

We observe

$$\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix}, \begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix}, \ldots, \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

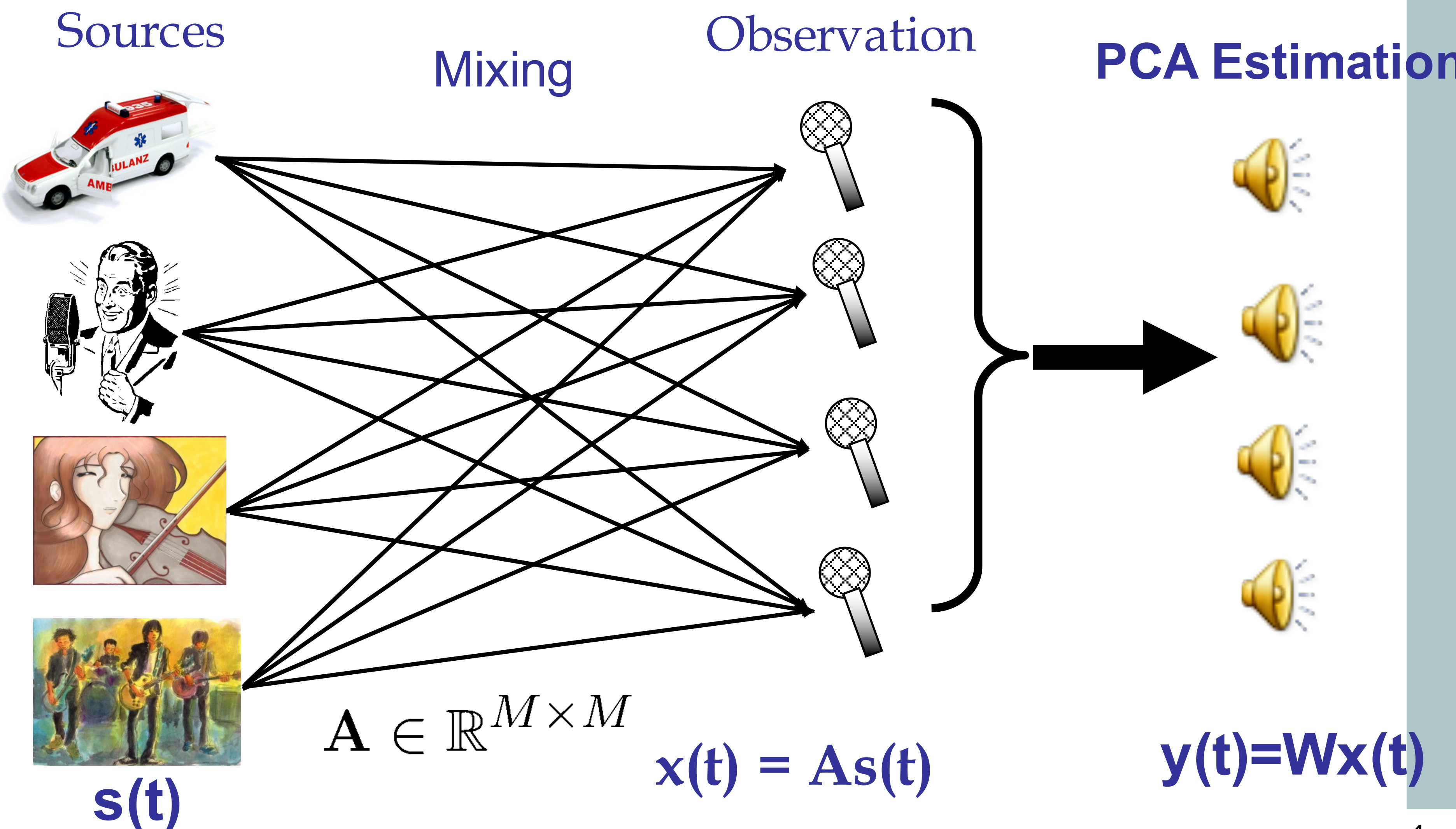
We want

$$\begin{pmatrix} s_1(1) \\ s_2(1) \end{pmatrix}, \begin{pmatrix} s_1(2) \\ s_2(2) \end{pmatrix}, \ldots, \begin{pmatrix} s_1(t) \\ s_2(t) \end{pmatrix}$$

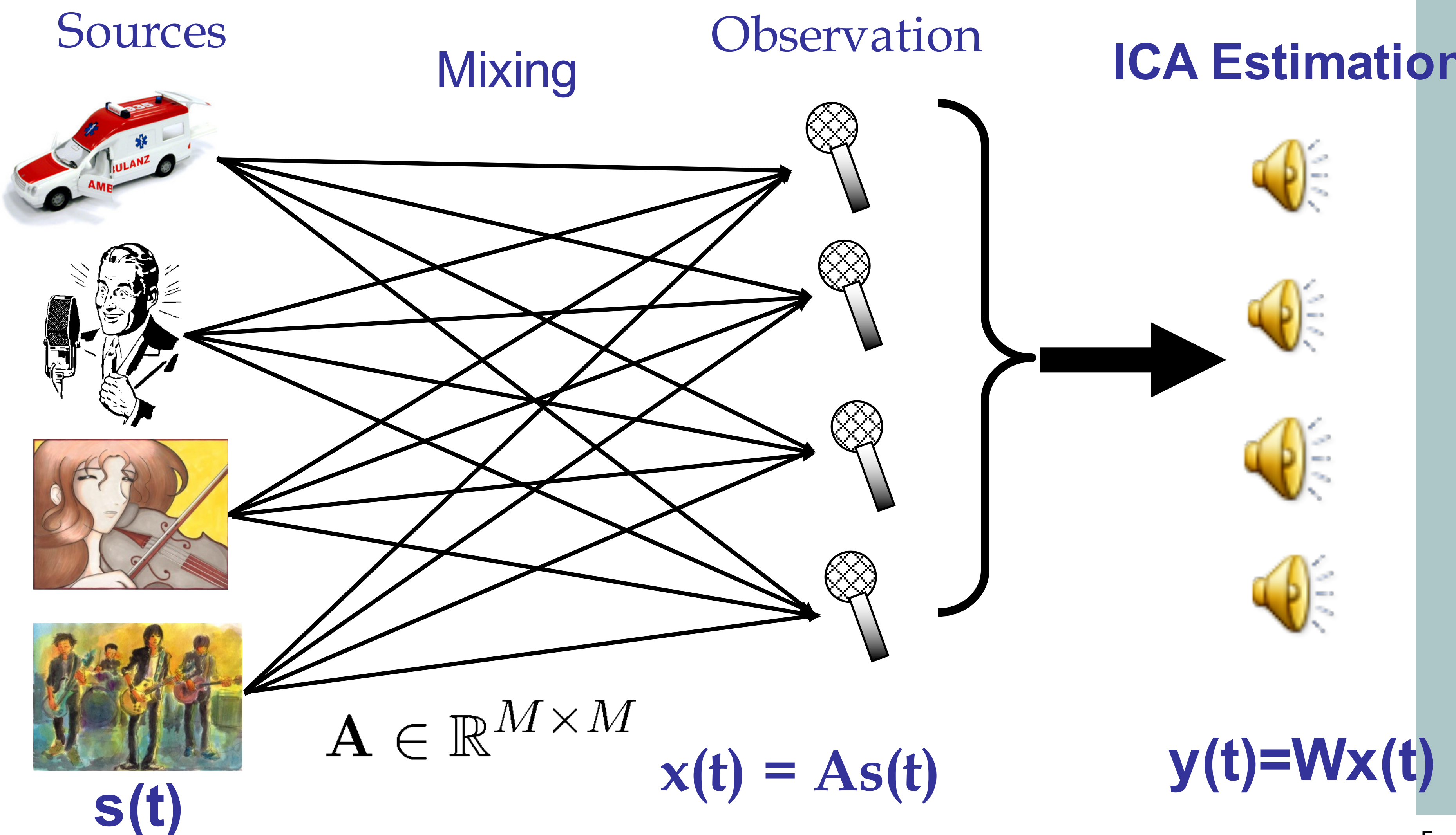
But we don't know $\{a_{ij}\}$, nor $\{s_i(t)\}$

Goal: Estimate $\{s_i(t)\}$, (and also $\{a_{ij}\}$)

The Cocktail Party Problem
SOLVING WITH PCA



The Cocktail Party Problem
SOLVING WITH ICA



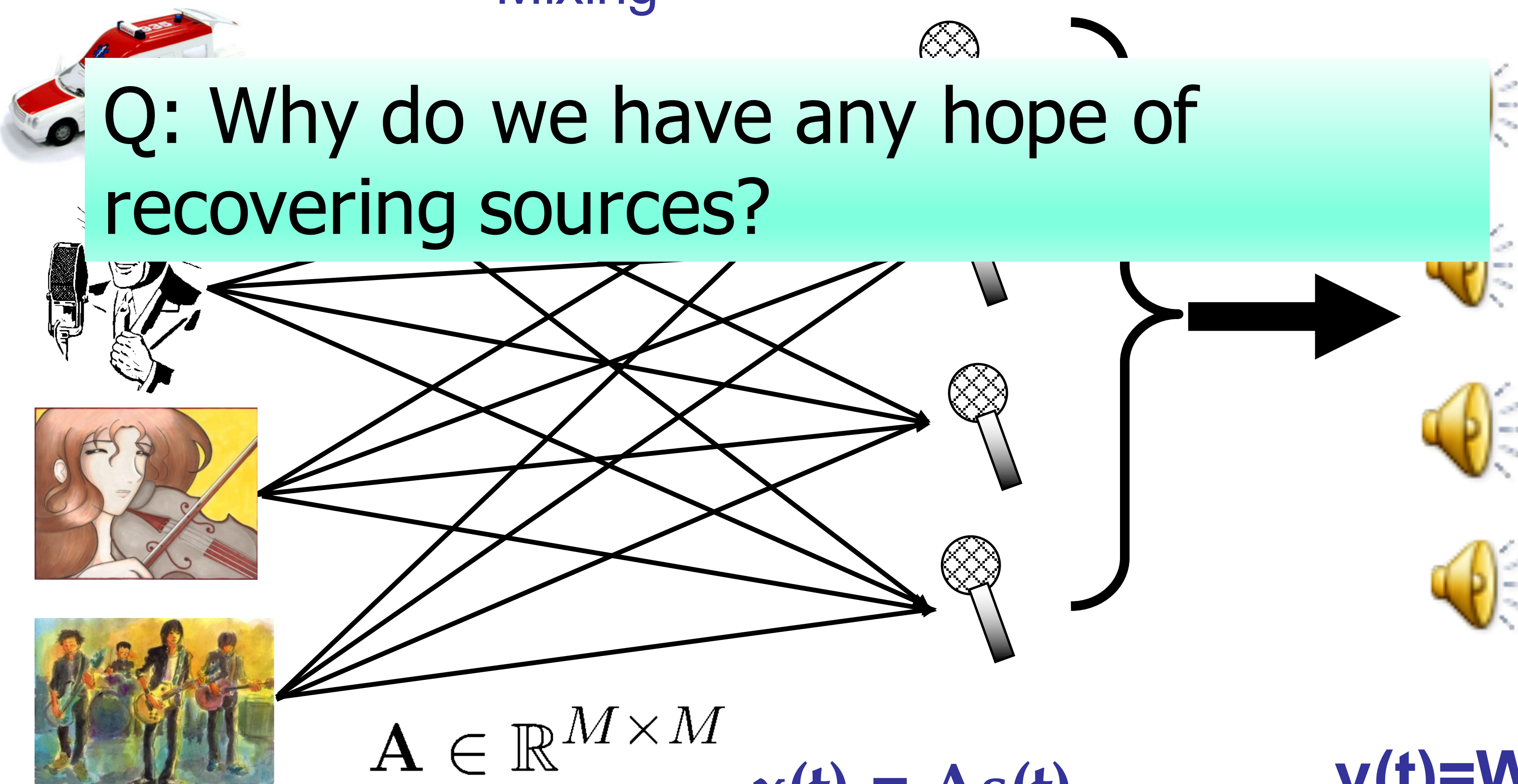
The Cocktail Party Problem
SOLVING WITH ICA

Sources

Mixing

Observation

ICA Estimation



Q: Why do we have any hope of recovering sources?

$\mathbf{A} \in \mathbb{R}^{M \times M}$

x(t) = As(t)

y(t)=Wx(t)

s(t)

The Cocktail Party Problem
SOLVING WITH ICA

Sources

Mixing

Observation

ICA Estimation

Q: Why do we have any hope of recovering sources?

A: Assume that sources are independent; find a linear transformation of data that is independent

$\mathbf{A} \in \mathbb{R}^{M \times M}$

$\mathbf{x(t) = As(t)}$

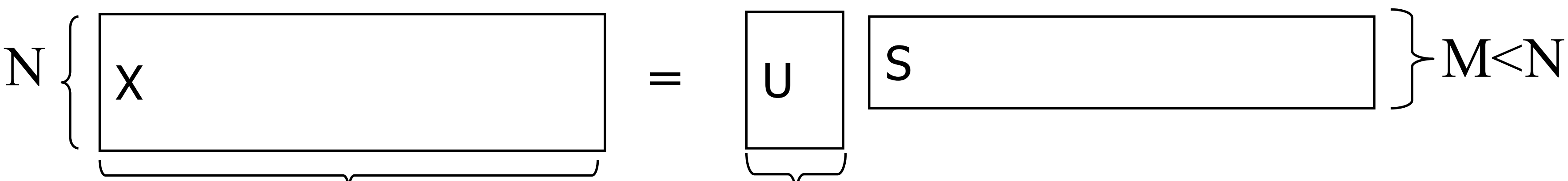
$\mathbf{y(t)=Wx(t)}$

$\mathbf{s(t)}$

ICA vs PCA

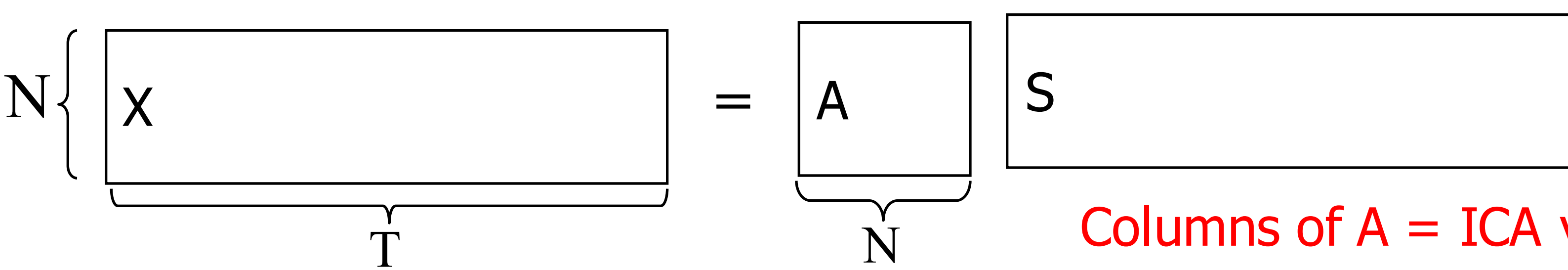
- Perform linear transformations
- Matrix factorization

PCA: ***low rank*** matrix factorization for ***compression***



Columns of U = PCA vectors

ICA: ***full rank*** matrix factorization to ***remove dependency*** among the rows



Columns of A = ICA vectors

ICA vs PCA

- PCA: $\mathbf{X=US,\ U^TU=I}$
- ICA: **X=AS, A is invertible**

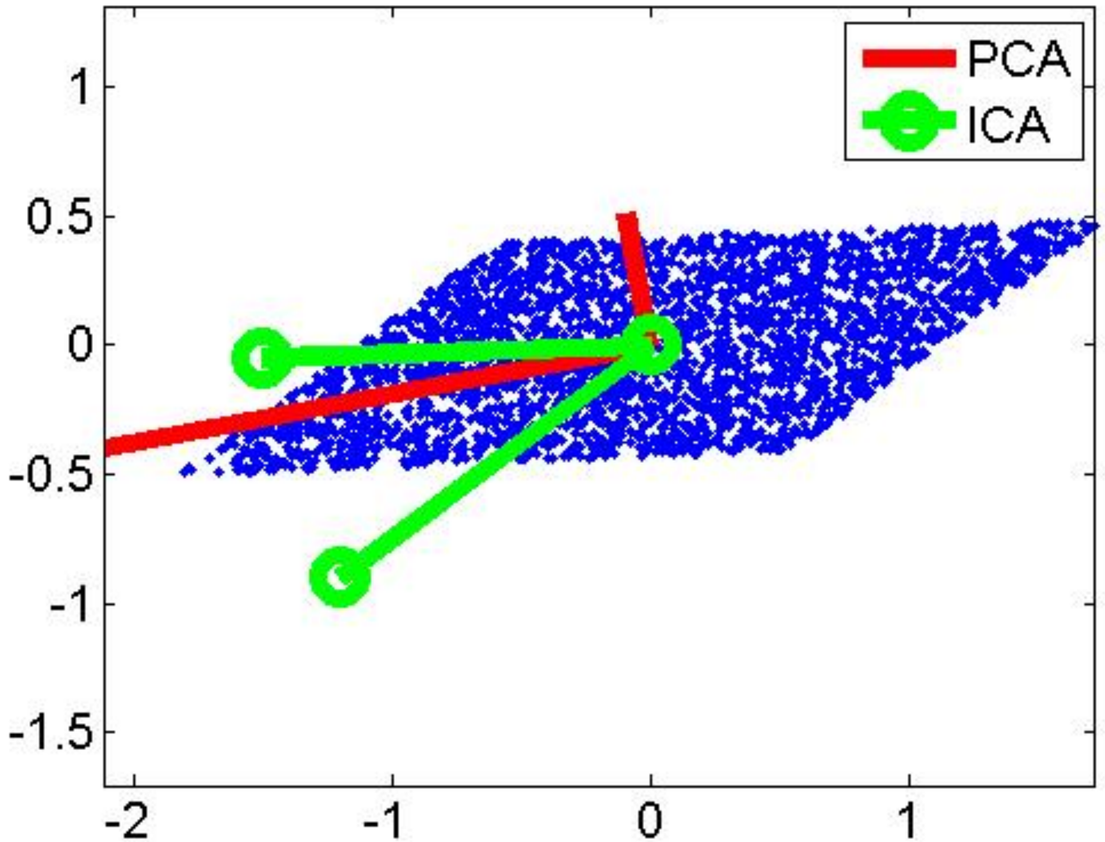
- PCA **does** compression
  - M<N

- ICA does **not** do compression
  - same # of features (M=N)

- PCA just removes correlations, **not** higher order dependence
- ICA removes correlations, **and** higher order dependence

- PCA: some components are **more important** than others (based on eigenvalues)
- ICA: components are **equally important**

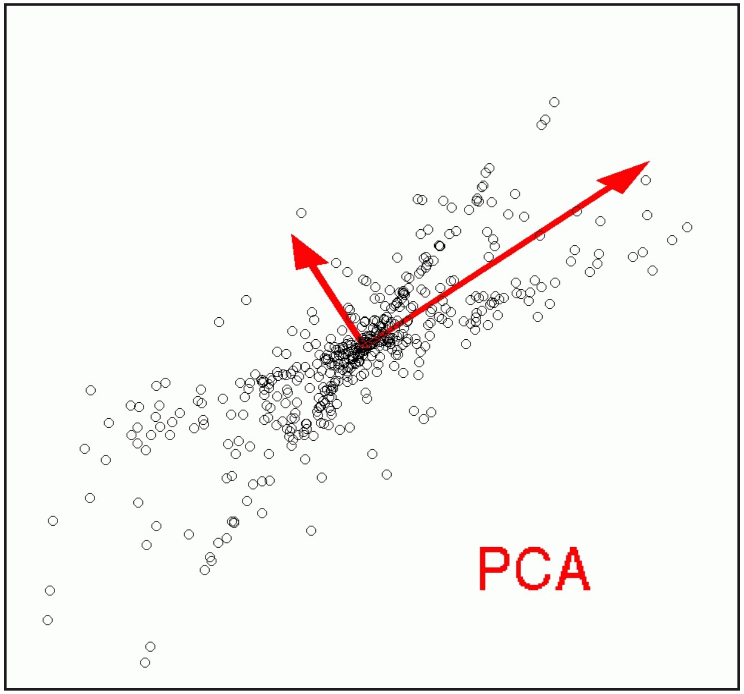
ICA vs PCA

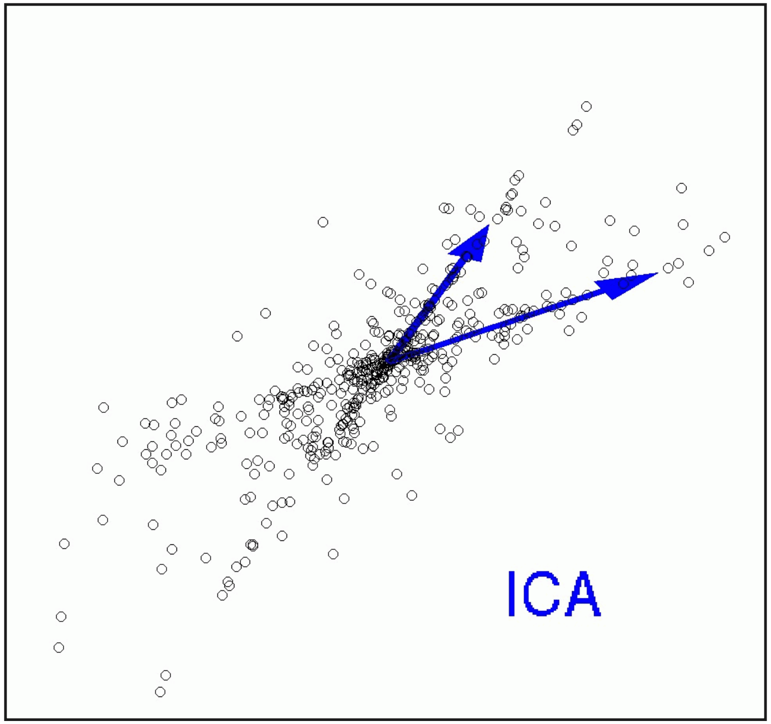


Note

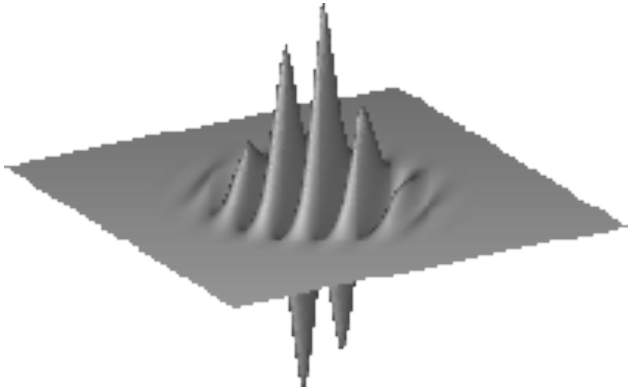
- PCA vectors are orthogonal
- ICA vectors are **not** orthogonal

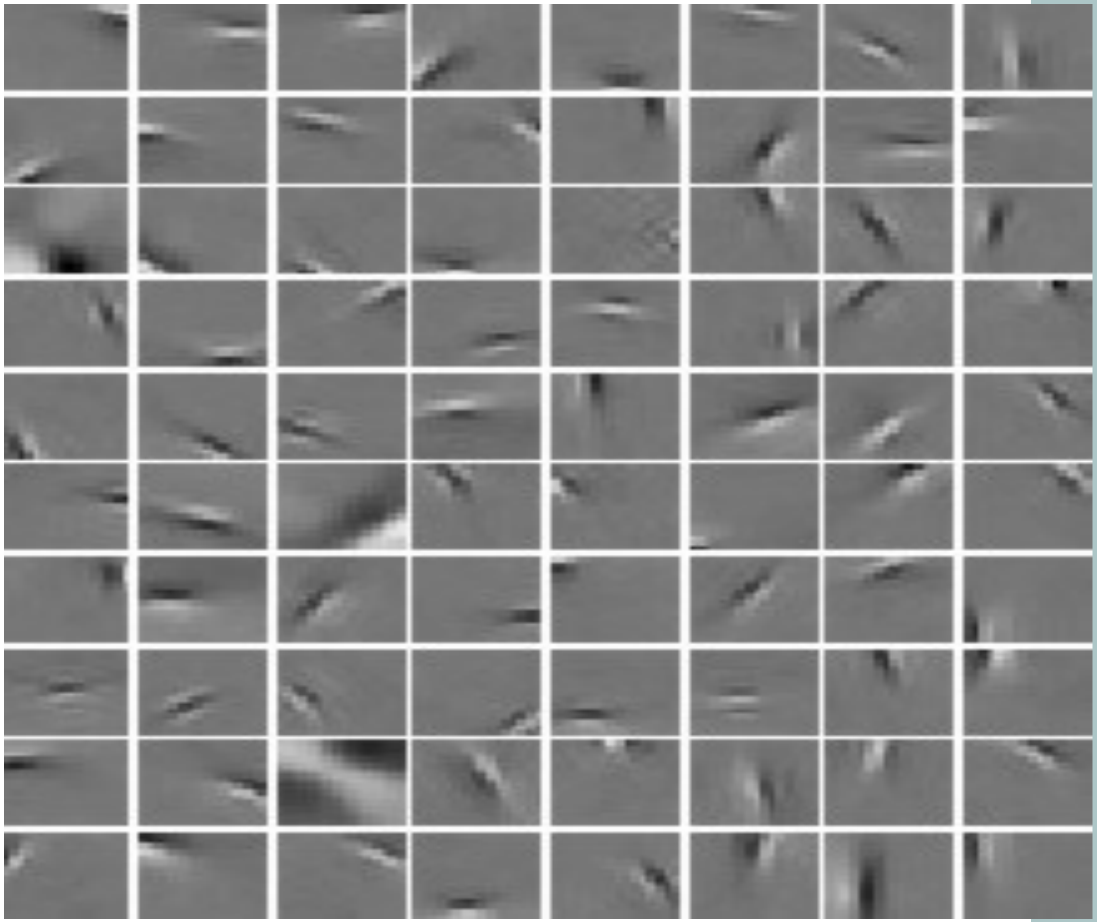
ICA vs PCA





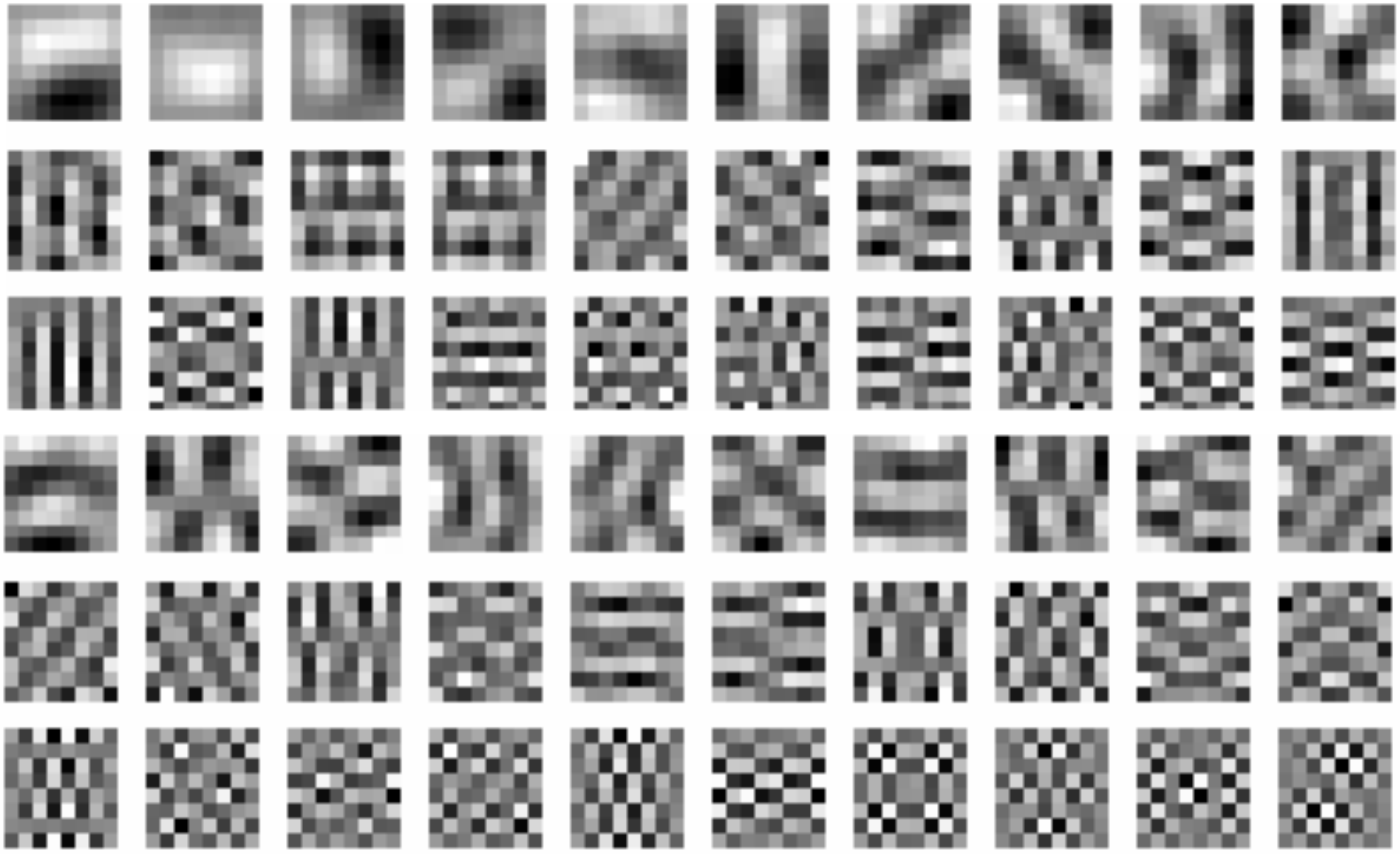
ICA basis vectors extracted from natural images





Gabor wavelets,
edge detection,
receptive fields of V1 cells..., deep neural networks

PCA basis vectors extracted from natural images



Some ICA Applications

STATIC

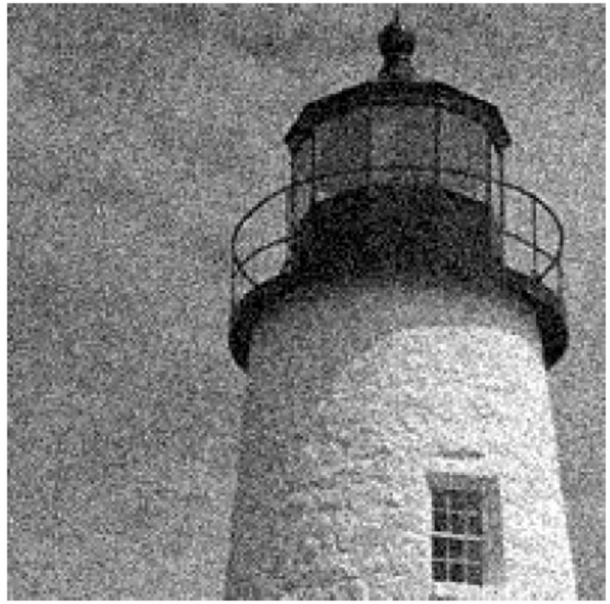
- Image denoising
- Microarray data processing
- Decomposing the spectra of galaxies
- Face recognition
- Facial expression recognition
- Feature extraction
- Clustering
- Classification
- Deep Neural Networks

TEMPORAL

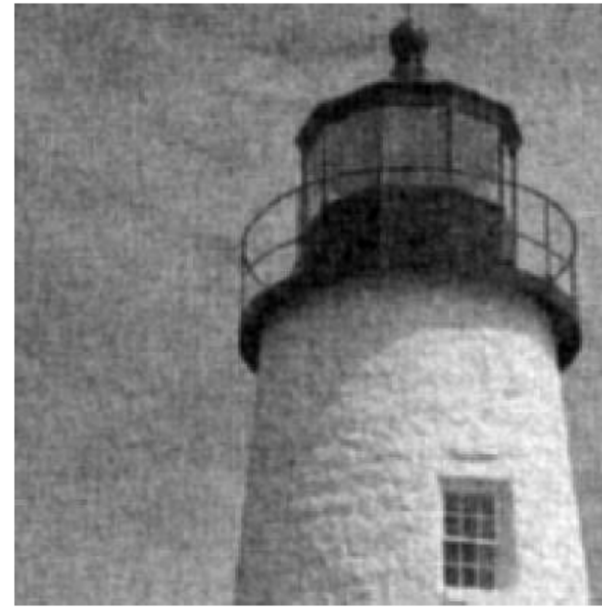
- Medical signal processing – fMRI, ECG, EEG
- Brain Computer Interfaces
- Modeling of the hippocampus, place cells
- Modeling of the visual cortex
- Time series analysis
- Financial applications
- Blind deconvolution

ICA for Image Denoising

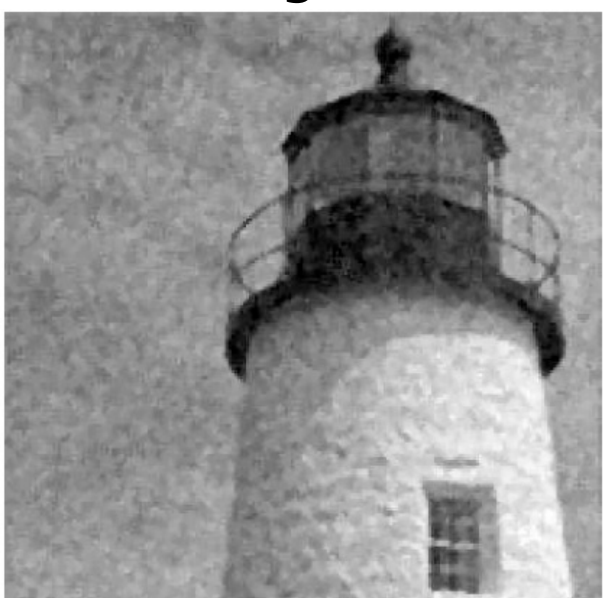
original



noisy



Wiener filtered



median filtered

ICA denoised

(Hoyer, Hyvarinen)

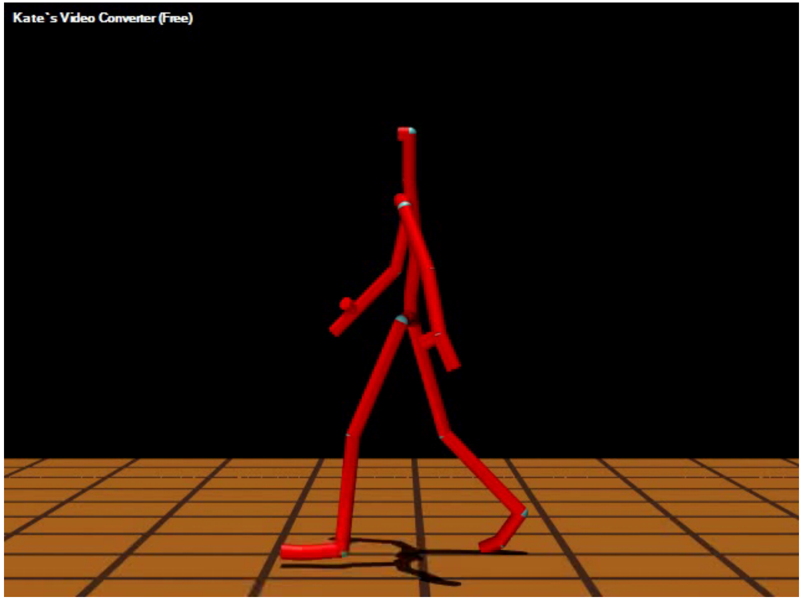
ICA for Motion Style Components

- Method for analysis and synthesis of human motion from motion captured data
- Provides perceptually meaningful “style” components
- 109 markers, (327dim data)
- Motion capture - data matrix

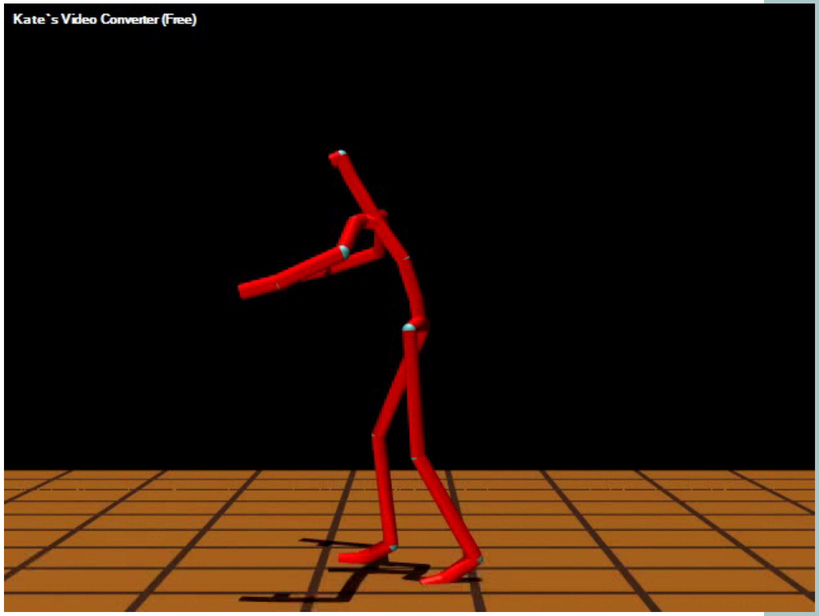
Goal: Find motion style components.

ICA - 6 independent components (emotion, content,…)

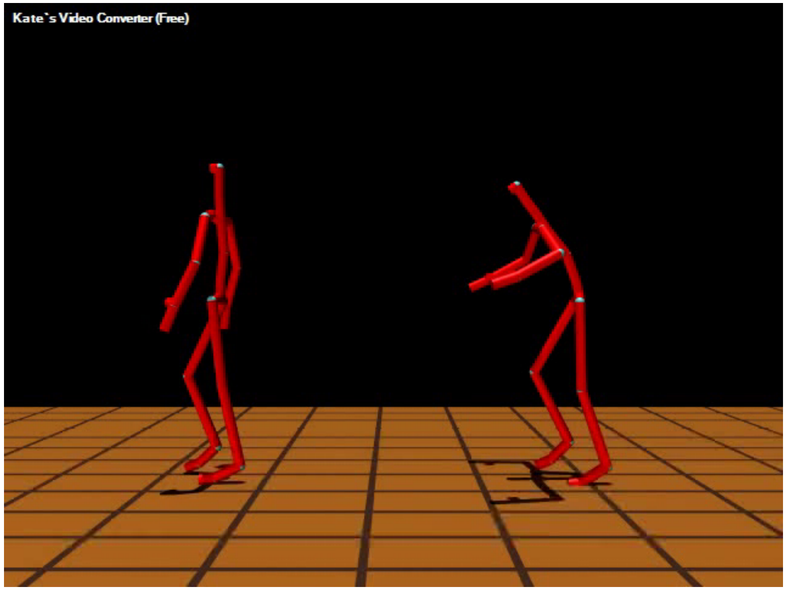
(Mori & Hoshino 2002, Shapiro et al 2006, Cao et al 2003)



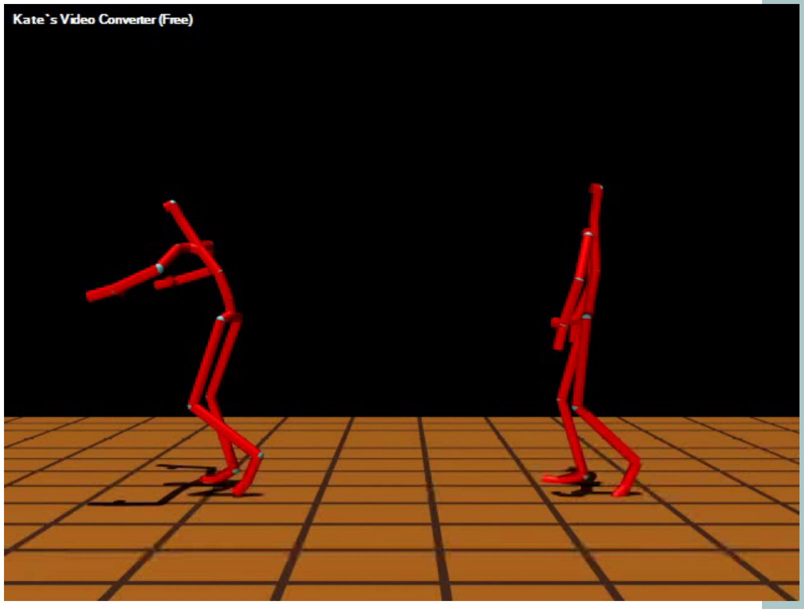
walk



sneaky



walk with sneaky



sneaky with walk

ICA Theory

Statistical (in)dependence

Definition (Independence)

Y_1, Y_2 are independent $\Leftrightarrow p(y_1, y_2) = p(y_1)\,p(y_2)$

Definition (Shannon entropy)

$$H(\mathbf{Y}) \doteq H(Y_1, \ldots, Y_m) \doteq -\int p(y_1, \ldots, y_m) \log p(y_1, \ldots, y_m) d\mathbf{y}.$$

Definition (Mutual Information) between more than 2 variables

$$0 \leq I(Y_1, \ldots, Y_M) \doteq \int p(y_1, \ldots, y_M) \log \frac{p(y_1, \ldots, y_M)}{p(y_1) \ldots p(y_M)} d\mathbf{y}$$

Definition (KL divergence)

$$0 \leq KL(f \| g) = \int f(x) \log \frac{f(x)}{g(x)} dx$$

Solving the ICA problem with i.i.d. sources

ICA problem: $\mathbf{x} = \mathbf{A}\mathbf{s}$, $\mathbf{s} = [s_1; \ldots; s_M]$ are jointly independent.

Ambiguity:

$\mathbf{s} = [s_1; \ldots; s_M]$ sources can be recovered only up to

sign, scale and permutation.

Proof:

- $\mathbf{P}$ = arbitrary permutation matrix,
- $\mathbf{\Lambda}$ = arbitrary diagonal scaling matrix.

$$\Rightarrow \mathbf{x} = [\mathbf{A}\mathbf{P}^{-1}\mathbf{\Lambda}^{-1}][\mathbf{\Lambda}\mathbf{P}\mathbf{s}]$$

Solving the ICA problem

Lemma:

We can assume that $E[\mathbf{s}] = 0$.

Proof:

Removing the mean does not change the mixing matrix.

$$\mathbf{x} - E[\mathbf{x}] = \mathbf{A}(\mathbf{s} - E[\mathbf{s}]).$$

In what follows we assume that $E[\mathbf{s}\mathbf{s}^T] = \mathbf{I}_M$, $E[\mathbf{s}] = 0$.

Whitening

- Let $\Sigma \doteq cov(\mathbf{x}) = E[\mathbf{x}\mathbf{x}^T] = \mathbf{A}E[\mathbf{s}\mathbf{s}^T]\mathbf{A}^T = \mathbf{A}\mathbf{A}^T$.

  (We assumed centered data)

- Do **SVD**: $\Sigma \in \mathbb{R}^{N\times N}$, $rank(\Sigma) = M$,

  $\Rightarrow \Sigma = \mathbf{U}\mathbf{D}\mathbf{U}^T$,

  where $\mathbf{U} \in \mathbb{R}^{N\times M}$, $\mathbf{U}^T\mathbf{U} = \mathbf{I}_M$, **Signular vectors**

  $\mathbf{D} \in \mathbb{R}^{M\times M}$, diagonal with rank M. **Singular values**

Whitening (continued)

- Let $\mathbf{Q} \doteq \mathbf{D}^{-1/2}\mathbf{U}^T \in \mathbb{R}^{M \times N}$ *whitening matrix*
- $\mathbf{x}^* \doteq \mathbf{Q}\mathbf{x}$

We have,

$$E[x^* {x^*}^T] = E[Q x x^T Q^T] = Q \Sigma Q^T = (D^{-1/2} U^T) U D U^T (U D^{-1/2}) = I_M$$

Whitening (continued)

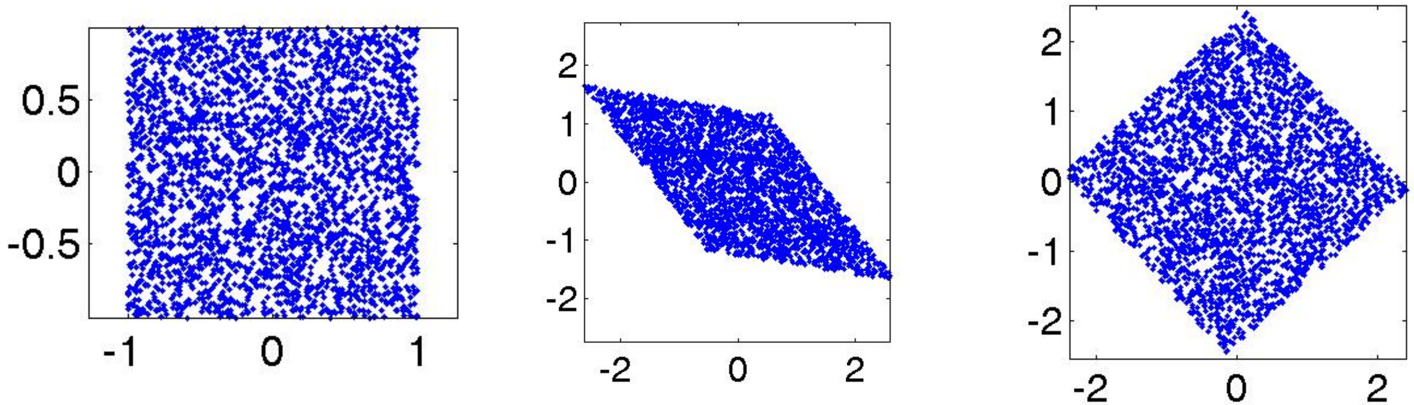
- Let $\mathbf{Q} \doteq \mathbf{D}^{-1/2}\mathbf{U}^T \in \mathbb{R}^{M \times N}$ *whitening matrix*
- Let $\mathbf{A}^* \doteq \mathbf{QA}$
- $\mathbf{x}^* \doteq \mathbf{Qx} = \mathbf{QAs} = \mathbf{A}^*\mathbf{s}$ is our new (*whitened*) ICA task.

We have,

$$E[x^* {x^*}^T] = E[Qxx^TQ^T] = Q\Sigma Q^T = (D^{-1/2}U^T)UDU^T(UD^{-1/2}) = I_M$$

$$\boxed{\Rightarrow E[\mathbf{x}^*{\mathbf{x}^*}^T] = \mathbf{I}_M, \text{ and } \mathbf{A}^*{\mathbf{A}^*}^T = \mathbf{I}_M.}$$

Whitening solves half of the ICA problem



original mixed whitened

After whitening it is enough to consider **orthogonal matrices** for separation.

Note:

The number of free parameters of an N by N orthonormal matrix is N(N-1)/2.

➔ whitening solves **half** of the ICA problem

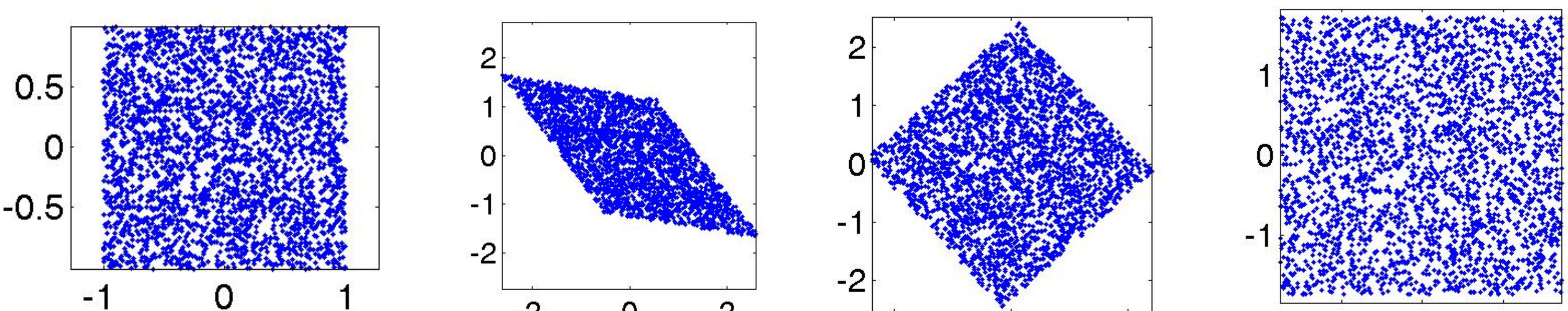
Solving ICA

ICA task: Given **x**,

- find **y** (the estimation of **s**),
- find **W** (the estimation of $\mathbf{A}^{-1}$)

ICA solution: **y=Wx**

- Remove mean, E[**x**]=0
- Whitening, $E[\mathbf{x}\mathbf{x}^T]=\mathbf{I}$
- Find an orthonormal **W** optimizing an objective function
  - Sequence of 2-d Jacobi (Givens) rotations



original mixed whitened rotated (demixed)

Optimization Using Jacobi Rotation Matrices

$$\mathbf{G}(p,q,\theta) \doteq \begin{array}{cc} \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \cos(\theta) & \cdots & -\sin(\theta) & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \sin(\theta) & \cdots & \cos(\theta) & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow p \\ \\ \leftarrow q \\ \\ \\ \end{array} \\ \begin{array}{ccccccc} \quad & \quad & \uparrow p & \qquad & \uparrow q & \quad & \quad \end{array} & \end{array} \in \mathbf{R}^{M\times M}$$

Observation : $\mathbf{x} = \mathbf{A}\mathbf{s}$

Estimation : $\mathbf{y} = \mathbf{W}\mathbf{x}$

$$\mathbf{W} = \arg\min_{\tilde{\mathbf{W}} \in \mathcal{W}} J(\tilde{\mathbf{W}}\mathbf{x}),$$

$$\text{where } \mathcal{W} = \{\mathbf{W} | \mathbf{W} = \prod_i G(p_i, q_i, \theta_i)\}$$

ICA Cost Functions

Let $\mathbf{y} \doteq \mathbf{W}\mathbf{x}$, $\mathbf{y} = [y_1; \ldots; y_M]$, and let us measure the dependence using Shannon's mututal information:

$$J_{ICA_1}(\mathbf{W}) \doteq I(y_1, \ldots, y_M) \doteq \int p(y_1, \ldots, y_M) \log \frac{p(y_1,\ldots,y_M)}{p(y_1)\ldots p(y_M)} d\mathbf{y},$$

Let $H(\mathbf{y}) \doteq H(y_1, \ldots, y_m) \doteq -\int p(y_1, \ldots, y_m) \log p(y_1, \ldots, y_m) d\mathbf{y}$.

Lemma

$$H(\mathbf{W}\mathbf{x}) = H(\mathbf{x}) + \log|\det \mathbf{W}|$$

Proof: Recitation

Therefore,

$$\begin{aligned} I(y_1, \ldots, y_M) &= \int p(y_1, \ldots, y_M) \log \frac{p(y_1, \ldots, y_M)}{p(y_1) \ldots p(y_M)} \\ &= -H(y_1, \ldots, y_M) + H(y_1) + \ldots + H(y_M) \\ &= -H(x_1, \ldots, x_M) - \log|\det \mathbf{W}| + H(y_1) + \ldots + H(y_M). \end{aligned}$$

ICA Cost Functions

$$
\begin{aligned}
I(y_1, \ldots, y_M) &= \int p(y_1, \ldots, y_M) \log \frac{p(y_1, \ldots, y_M)}{p(y_1) \ldots p(y_M)} \\
&= -H(y_1, \ldots, y_M) + H(y_1) + \ldots + H(y_M) \\
&= -H(x_1, \ldots, x_M) - \log |\det \mathbf{W}| + H(y_1) + \ldots + H(y_M).
\end{aligned}
$$

$H(x_1, \ldots, x_M)$ is constant

Does not depend on **W**

$$\log |\det \mathbf{W}| = 0.$$

W is a product of Givens rotations

$$\mathbf{W} = \prod_i G(p_i, q_i, \theta_i)$$

$$\det(\mathbf{W}) = \prod_i \det(G(p_i, q_i, \theta_i)) = 1$$

ICA Cost Functions

$$
\begin{aligned}
I(y_1,\ldots,y_M) &= \int p(y_1,\ldots,y_M)\log\frac{p(y_1,\ldots,y_M)}{p(y_1)\ldots p(y_M)} \\
&= -H(y_1,\ldots,y_M)+H(y_1)+\ldots+H(y_M) \\
&= -H(x_1,\ldots,x_M)-\log|\det\mathbf{W}|+H(y_1)+\ldots+H(y_M).
\end{aligned}
$$

$H(x_1,\ldots,x_M)$ is constant, $\log|\det\mathbf{W}|=0.$

Therefore,

$$\boxed{J_{ICA_2}(\mathbf{W}) \doteq H(y_1)+\ldots+H(y_M)}$$

$\mathbf{E[yy^T] = WE[zz^T]W^T = WW^T = I}$
(since W is orthogonal)

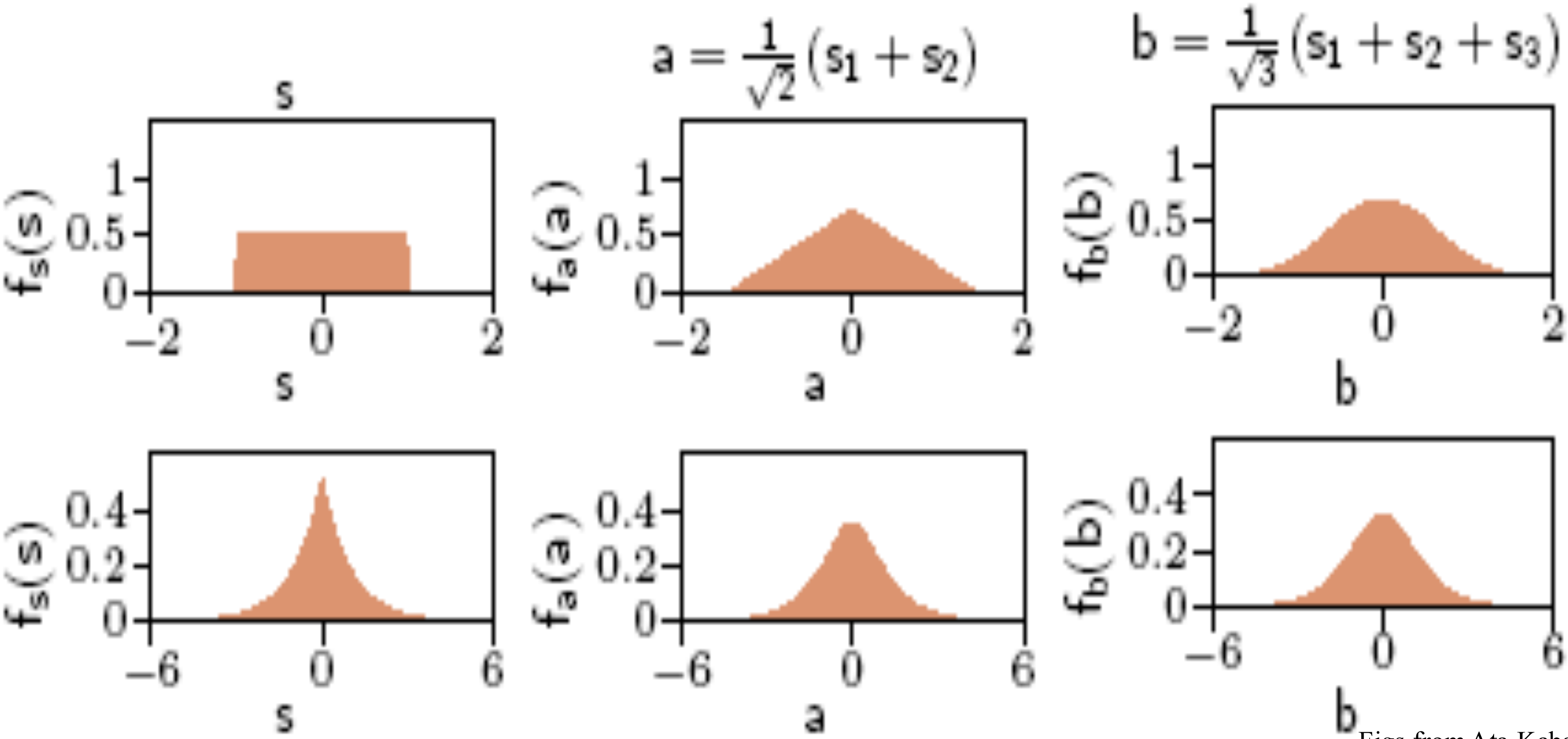
The covariance is fixed: Which distribution has the least entropy when fixing covariance?

Normal distribution has the most entropy: so "least normal" distribution

Central Limit Theorem

The sum of independent variables converges to the normal distribution

➔ For separation go far away from the normal distribution



Figs from Ata Kaban

ICA Algorithms

Minimizing sum of entropy is hard since it requires knowing density

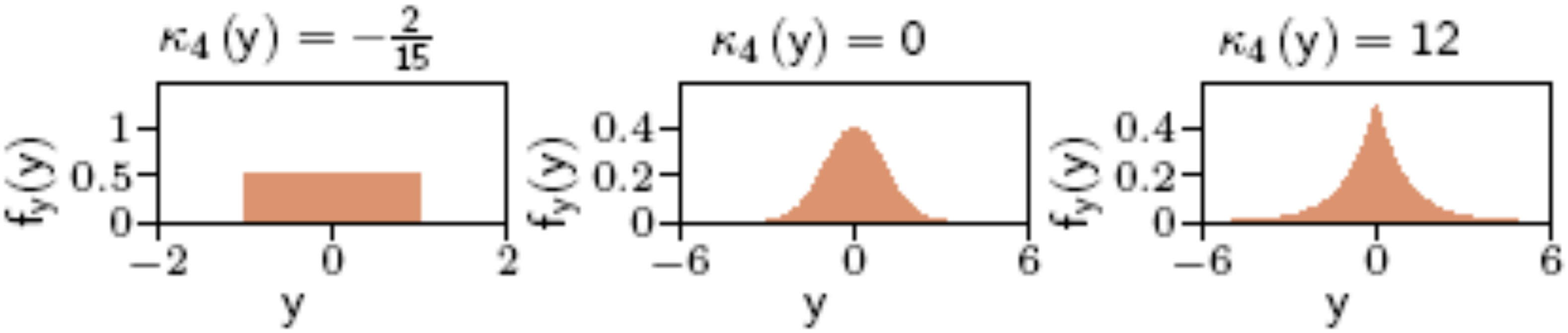
ICA algorithms use other measures of non-Gaussianity

ICA algorithm based on Kurtosis maximization

Kurtosis = 4th order cumulant

Measures degree of peakedness

- $\kappa_4(y) = \mathrm{E}\{y^4\} - \underbrace{3\left(\mathrm{E}\{y^2\}\right)^2}_{=3 \text{ if } \mathrm{E}\{y\}=0 \text{ and whitened}}$



The Fast ICA algorithm (Hyvarinen)

- Given whitened data $\mathbf{z}$
- Estimate the 1^{st} ICA component:

Probably the most famous ICA algorithm

$\star\ y = \mathbf{w}^T \mathbf{z},\ \|\mathbf{w}\| = 1, \qquad \Leftarrow \mathbf{w}^T = 1^{st}$ row of $\mathbf{W}$

$$\boxed{\begin{aligned}&\star \text{ maximize kurtosis } f(\mathbf{w}) \doteq \kappa_4(y) \doteq \mathbb{E}[y^4]\text{-}3\\ &\qquad\qquad \text{with constraint } h(\mathbf{w}) = \|\mathbf{w}\|^2 - 1 = 0\end{aligned}}$$

$\star$ At optimum $f'(\mathbf{w}) + \lambda h'(\mathbf{w}) = \mathbf{0}^T$ (λ Lagrange multiplier)

$$\Rightarrow 4\mathbb{E}[(\mathbf{w}^T\mathbf{z})^3\mathbf{z}] + 2\lambda\mathbf{w} = 0$$

Solve this equation by Newton–Raphson's method.

Newton method for finding a root

Newton Method for Finding a Root

Goal: $\phi : \mathbb{R} \to \mathbb{R}$

$$\phi(x^*) = 0$$

$$x^* = ?$$

Linear Approximation (1st order Taylor approx**):**

$$\underbrace{\phi(\underbrace{x + \Delta x}_{x^*})}_{\phi(x^*) = 0} = \phi(x) + \phi'(x)\Delta x + o(|\Delta x|)$$

Therefore,

$$0 \approx \phi(x) + \phi'(x)\Delta x$$

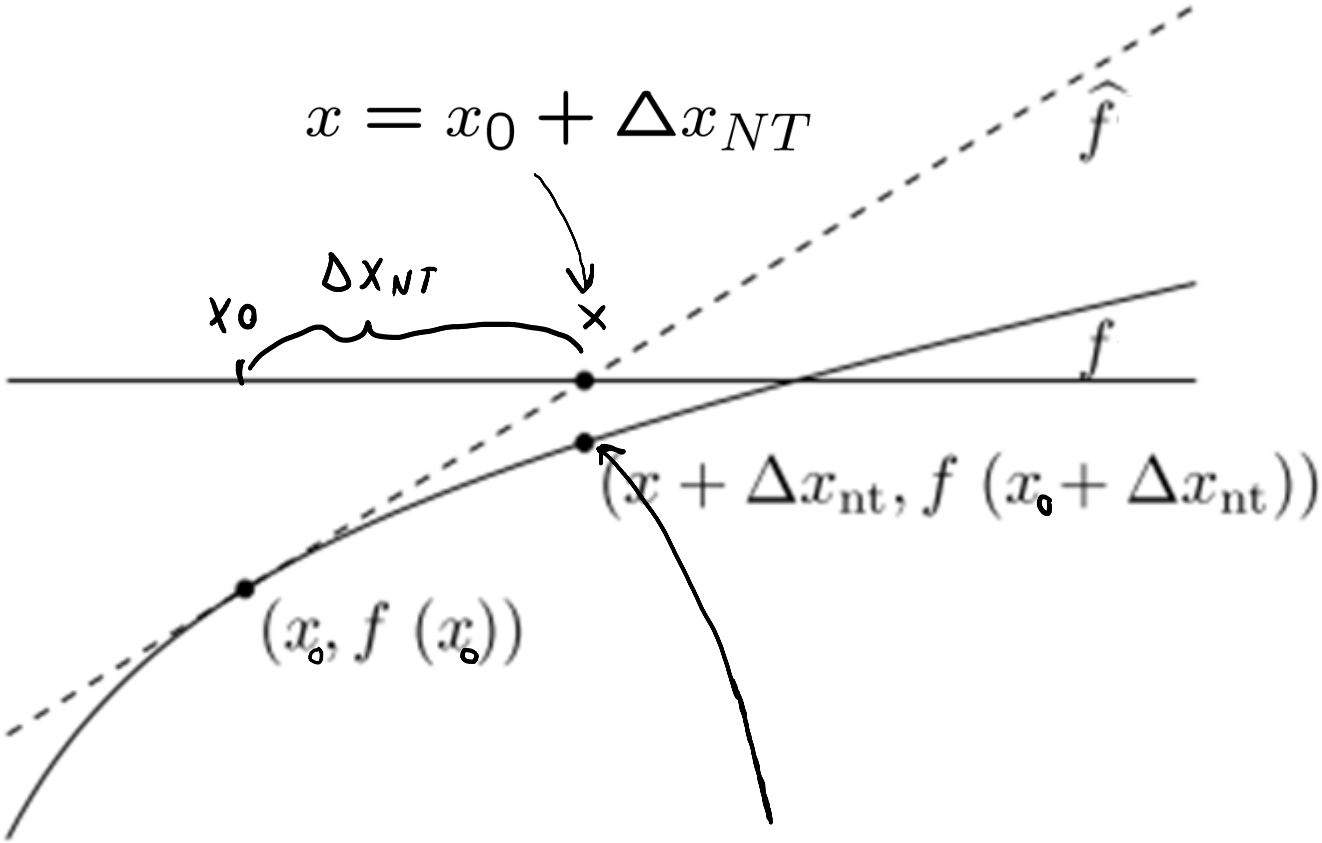
$$x^* - x = \Delta x = -\frac{\phi(x)}{\phi'(x)}$$

$$x_{k+1} = x_k - \frac{\phi(x)}{\phi'(x)}$$

Illustration of Newton's method

Goal: finding a root

$$\widehat{f}(x) = f(x_0) + f'(x_0)(x - x_0)$$



In the next step we will linearize here in *x*

Newton Method for Finding a Root

This can be generalized to multivariate functions

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$0_m = F(x^*) = F(x + \Delta x) = F(x) + \underbrace{\nabla F(x)}_{\mathbb{R}^{m \times n}} \underbrace{\Delta x}_{\mathbb{R}^n} + o(|\Delta x|)$$

NEGLECT

Therefore,

$$0_m = F(x) + \nabla F(x) \Delta x$$

$$\Delta x = -[\nabla F(x)]^{-1} F(x)$$

[Pseudo inverse if there is no inverse]

$\Delta x = x_{k+1} - x_k$, and thus

$$\underbrace{x_{k+1}}_{\mathbb{R}^n} = \underbrace{x_k}_{\mathbb{R}^n} - \underbrace{[\nabla F(x_k)]^{-1}}_{\mathbb{R}^{n \times m}} \underbrace{F(x_k)}_{\mathbb{R}^m}$$

Newton method: Start from x_0 and iterate.

Newton method for FastICA

The Fast ICA algorithm (Hyvarinen)

Solve: $F(\mathbf{w}) = 4\mathbb{E}[(\mathbf{w}^T\mathbf{z})^3\mathbf{z}] + 2\lambda\mathbf{w} = 0$

Note:

$$y = \mathbf{w}^T\mathbf{z},\ \|\mathbf{w}\| = 1,\ \mathbf{z} \text{ white} \Rightarrow \mathbb{E}[(\mathbf{w}^T\mathbf{z})^2] = 1$$

$\mathbb{E}[z z^T] = I$

$\mathbb{E}[w^T z z^T w] = w^T I w = 1$

$\|w\| = 1$

The derivative of ***F*** **:**

$$F'(\mathbf{w}) = 12\mathbb{E}[(\mathbf{w}^T\mathbf{z})^2\mathbf{z}\mathbf{z}^T] + 2\lambda\mathbf{I}$$
$$\sim 12\mathbb{E}[(\mathbf{w}^T\mathbf{z})^2]\mathbb{E}[\mathbf{z}\mathbf{z}^T] + 2\lambda\mathbf{I}$$
$$= 12\mathbb{E}[(\mathbf{w}^T\mathbf{z})^2]\mathbf{I} + 2\lambda\mathbf{I}$$
$$= 12\mathbf{I} + 2\lambda\mathbf{I}$$

The Fast ICA algorithm (Hyvarinen)

The Jacobian matrix becomes diagonal, and can easily be inverted.

$$\mathbf{w}(k+1) = \mathbf{w}(k) - [F'(\mathbf{w}(k)]^{-1} F(\mathbf{w}(k))$$

$$\mathbf{w}(k+1) = \mathbf{w}(k) - \frac{4\mathbb{E}[(\mathbf{w}(k)^T\mathbf{z})^3\mathbf{z}]+2\lambda\mathbf{w}(k)}{12+2\lambda}$$

$$(12+2\lambda)\mathbf{w}(k+1) = (12+2\lambda)\mathbf{w}(k) - 4\mathbb{E}[(\mathbf{w}(k)^T\mathbf{z})^3\mathbf{z}] - 2\lambda\mathbf{w}(k)$$

$$-\frac{12+2\lambda}{4}\mathbf{w}(k+1) = -3\mathbf{w}(k) + \mathbb{E}[(\mathbf{w}(k)^T\mathbf{z})^3\mathbf{z}]$$

Therefore,

Let $\mathbf{w}_1$ be the fix pont of:

$$\tilde{\mathbf{w}}(k+1) = \mathbb{E}[(\mathbf{w}(k)^T\mathbf{z})^3\mathbf{z}] - 3\mathbf{w}(k)$$

$$\mathbf{w}(k+1) = \frac{\tilde{\mathbf{w}}(k+1)}{\|\tilde{\mathbf{w}}(k+1)\|}$$

- Estimate the 2^{nd} ICA component similarly using the $\mathbf{w} \perp \mathbf{w}_1$ additional constraint... and so on ...

What you should know

ICA: **x** = **As** matrix factorization to identify independent components

PCA vs ICA – correlation vs general dependence

ICA:

Given **x**, remove mean and whiten to get **z**

Find **W** (the estimation of $\mathbf{A}^{-1}$)

orthogonal matrix – product of 2d Givens rotations

obtained by maximizing non-Gaussianity

Find **y** = **Wz** (the estimation of **s**)

ICA algorithms:

Kurtosis Maximization

FastICA

Maximum Likelihood ICA Algorithm

David J.C. MacKay (97)

- simplest approach
- requires knowing densities of hidden sources $\{f_i\}$

rows of **W**

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t),\ \mathbf{s}(t) = \mathbf{W}\mathbf{x}(t),\ \text{where } \mathbf{A}^{-1} = \mathbf{W} = [\mathbf{w}_1; \ldots; \mathbf{w}_M] \in \mathbb{R}^{M\times M}$$

$$L = \sum_{t=1}^{T} \log P_x(x(t)) = \sum_{t=1}^{T} \log \underbrace{P_x(As(t))}_{A^{-1}P_s(\underbrace{A^{-1}x(t)}_{s(t)})} \Rightarrow \max_A$$

$$P_{As}(u) = A^{-1} P_s(A^{-1}u)$$

$$\Rightarrow L = \sum_{t=1}^{T} \log A^{-1} P_s(s(t)) = T\log|W| + \sum_{t=1}^{T} \log \underbrace{P_s(s(t))}_{\prod_{i=1}^{M} P_{s_i}(s_i(t))}$$

$$= T\log|W| + \sum_{t=1}^{T}\sum_{i=1}^{M} \log \underbrace{P_{s_i}(w_i x(t))}_{f_i(w_i x(t))}$$

$$\Rightarrow \max_W$$

Maximum Likelihood ICA Algorithm

$$L = T \log |W| + \sum_{t=1}^{T} \sum_{k=1}^{M} \log f_k(w_k x(t))$$

$$\Rightarrow \max_{W} L \Rightarrow \frac{\partial L}{\partial w_{ij}} = ?$$

$$\frac{\partial L}{\partial w_{ij}} = T(W^T)^{-1}_{ij} + \sum_{t=1}^{T} \underbrace{\frac{\partial}{\partial w_{ij}} \sum_{k=1}^{M} \log f_k(w_k x(t))}_{\frac{f_i'(w_i x(t))}{f_i(w_i x(t))} x_j(t)}$$

$$= T(W^T)^{-1}_{ij} + \sum_{t=1}^{T} \frac{f_i'(w_i x(t))}{f_i(w_i x(t))} x_j(t)$$

$$\Rightarrow \boxed{\Delta \mathbf{W} \propto [\mathbf{W}^T]^{-1} + \frac{1}{T} \sum_{t=1}^{T} g(\mathbf{W}\mathbf{x}(t))\mathbf{x}^T(t),}$$ where $g_i = f_i'/f_i$