Systems of Nonlinear Equations

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Nonlinear Systems

One dimensional:

$f : \mathrm{R} \rightarrow \mathrm{R}$

e.g. $f(x) = ax\text{-}b = 0$ (linear)

or $f(x) = \sin(2x)\text{-}x = 0$ (nonlinear)

Multidimensional

$f : \mathrm{R}^n \rightarrow \mathrm{R}^n$

e.g. $f(x) = Ax\text{-}b = 0$ (linear system)

or $\{x^2+y^2+xy\text{-}1=0,\ x^2+y^2\text{-}xy\text{-}1=0\}$ (nonlinear: two ellipses)

also known as root finding

Four 1-D Root-Finding Methods

- Bisection Method
- Newton's Method
- Secant Method
- Linear Fractional Interpolation

Bisection Method (Binary Search)

Given $f()$, tol, and $[a,b]$ such that $\text{sign}(f(a)) \neq \text{sign}(f(b))$:

while b-a>tol

$m = (a+b)/2$ midpoint

if $\text{sign}(f(a)) = \text{sign}(f(m))$: then

$a = m$ recurse on right half

else

$b = m$ recurse on left half

Guaranteed convergence! (if you can find initial a,b)

but only at a linear rate: $|\text{error}_{k+1}| \leq c|\text{error}_k|$, $c<1$

Newton's Method

Given $f()$, $f'()$, tol, and initial guess x_0

$k = 0$

do

  $x_{k+1} = x_k - f(x_k)/f'(x_k)$

  k++

while $x_k - x_{k-1} >$ tol

Can diverge (especially if $f' \approx 0$).

If it converges, does so at quadratic rate: $|e_{k+1}| \leq c|e_k|^2$.

Requires derivative computation.

Secant Method

Like Newton, but approximates slope using point pairs

Given *f()*, tol, and initial guesses x_0, x_1

$k = 1$

do

$x_{k+1} = x_k - f(x_k)(x_k - x_{k-1})/(f(x_k)-f(x_{k-1}))$

k++

while $x_k - x_{k-1} < \text{tol}$

Can diverge.

If it converges, does so at rate 1.618: $|e_{k+1}| \leq c|e_k|^{1.618}$.

Linear Fractional Interpolation

Instead of linear approximation, fit with fractional linear approximation: $f(x) \approx (x-u)/(vx-w)$ for some u,v,w

Given $f()$, tol, and initial guesses a, b, c, $f_a=f(a)$, $f_b=f(b)$

do

$h = (a-c)(b-c)(f_a-f_b)f_c \,/\, [(a-c)(f_c-f_b)f_a-(b-c)(f_c-f_a)f_b]$

$a = b;\ \ f_a = f_b$

$b = c;\ \ f_b = f_c$

$c \mathrel{+}= h;\ \ f_c = f(c)$

while $h >$ tol

If it converges, does so at rate 1.839: $|e_{k+1}| \le c|e_k|^{1.839}$.

1-D Root-Finding Summary

- Bisection: safe: never diverges, but slow.
- Newton’s method: risky but fast!
- Secant method & linear fractional interpolation : less risky, mid-speed.

- Hybrids: use a safe method where function poorly behaved, Newton’s method where it’s well-behaved.

Multidimensional Newton's Method

f and x are n-vectors

First order Taylor series approx.: $f(x+h) \approx f(x)+f'(x)h$

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad J_f(x) = f'(x) = \begin{bmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial x_1 & \cdots & \partial f_n/\partial x_n \end{bmatrix}$$

$f(x+h)=0$ implies $h=-J^{-1}f$ (don't compute J^{-1}, but solve $Jh=f$)

so iteration should be

$$x_{k+1} = x_k - J^{-1}(x_k)f(x_k)$$

Method requires calculation of Jacobian – can be expensive

Example: Intersection of Ellipses

2 equations in 2 unknowns:

$$x^2+y^2+xy-1=0$$

$$x^2+y^2-xy-1=0$$