A Theorist's Toolkit **(CMU 18-859T, Fall 2013)**

Lecture 13: Linear programming I

October 21, 2013

Lecturer: Ryan O'Donnell *Scribe: Yuting Ge*

A Generic Reference: Ryan recommends a very nice book "Understanding and Using Linear Programming" [MG06].

1 Introduction

Linear Programming (**LP**) refers to the following problem. We are given an input of the following m constraints (inequalities):

$$K \subseteq R^n = \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \end{cases}$$

where every $a_{ij}, b_i \in \mathbb{Q}$, for $i \in [m], j \in [n]$. Our goal is to determine whether a solution exists, i.e., $K \neq \varnothing$, or to maximize $c_1x_1 + \cdots + c_nx_n$ for some $c_1 \cdots c_n \in \mathbb{Q}$ (we work toward the first goal for now; we'll talk about the second goal later). If $K \neq \varnothing$ we output a point in $\mathbb{Q}^n$ in K; if $K = \varnothing$ we output a "proof" that it's empty. We'll prove that if $K \neq \varnothing$ we can always find a rational solution x^* (i.e., $x^* \in \mathbb{Q}^n$) in K, and define what a proof of unsatisfiability is in the next section. A remark of this problem:

Remark 1.1. We can change the "$\geq$" in the definition to "$\leq$" since $a \cdot x \leq b \iff -a \cdot x \geq -b$. We can also allow equality since $a \cdot x = b \iff a \cdot x \geq b \wedge a \cdot x \leq b$. However, we CANNOT allow "$>$" or "$<$".

The significance of this problem is:

Theorem 1.2. *[Kha79]* ***LP*** *is solvable in polynomial-time.*

2 Fourier-Motzkin Elimination

If some (non-negative) linear combinations of the m input constraints give us $0 \geq 1$, then clearly the LP is unsatisfiable (we emphasize non-negative so that we don't change the direction of inequality). Thus we define:

Definition 2.1. A proof of unsatisfiability is m multipliers $\lambda_1 \cdots \lambda_m \geq 0$ such that the sum over $i \in [m]$ of λ_i times the i^{th} constraint gives us $0 \geq 1$, i.e.,

$$\begin{cases} \lambda_1 a_{1i} + \lambda_2 a_{2i} + \cdots + \lambda_m a_{mi} = n & \text{for all } i \in [n] \\ \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_m b_m = 1 \end{cases}$$

Here's a few remarks to this definition:

Remark 2.2. We don't need $0 \geq 1$; $0 \geq q$ for any positive q would suffice. However we may divide every λ_i by q to get $0 \geq 1$.

Remark 2.3. If our LP algorithm outputs a solution for $K \neq \varnothing$ but outputs nothing for $K = \varnothing$, we can use this algorithm to generate the $\lambda_i's$, since we can see from our definition that the existence of these $\lambda_i's$ is an LP (with m variables and $n+1$ constraints).

It turns out that if an LP is unsatisfiable, a proof exists.

Theorem 2.4 (Farkas Lemma/LP duality)**.** *$K \neq \varnothing \implies$ such λ_i's exist.*

We'll "prove" the result with an example of Fourier-Motzkin Elimination.

Proof. With loss of generality (WLOG), assume we are working with a 3-variable 5-constraint LP, namely,

$$\begin{cases} x - 5y + 2z \geq 7 \\ 3x - 2y - 6z \geq -12 \\ -2x + 5y - 4z \geq -10 \\ -3x + 6y - 3z \geq -9 \\ -10y + z \geq -15 \end{cases}$$

Our first step is to eliminate x. We do this by multiplying each constraint with a positive constant such that the coefficient of x in each constraint is either -1, 1, or 0 (if an original constraint has 0 as the coefficient of x we just leave it alone). In our example, we get:

$$\begin{cases} x - 5y + 2z \geq 7 \\ x - \frac{2}{3}y - 2z \geq -4 \\ -x + \frac{5}{2}y - 2z \geq -5 \\ -x + 2y - z \geq -3 \\ -10y + z \geq -15 \end{cases}$$

We then write the inequalities as $x \geq c_i y + d_i z + e_i$ or $x \leq c_i y + d_i z + e_i$ for some $c_i, d_i, e_i \in \mathbb{Q}$, depending on whether x has coefficient 1 or -1 (again, if x has coefficient 0 we just leave it alone) in the inequality, i.e.,

$$\begin{cases} x \geq 5y - 2z + 7 \\ x \geq \frac{2}{3}y + 2z - 4 \\ x \leq \frac{5}{2}y - 2z + 5 \\ x \leq 2y - z + 3 \\ -10y + z \geq -15 \end{cases}$$

In order to satisfy the inequality, for each pair $x \geq c_i y + d_i z + e_i, x \leq c_j y + d_j z + e_j$, we must have $c_j y + d_j z + e_j \geq c_i y + d_i z + e_i$. Without changing the satisfiability of the original constraints, we change them (again, we leave alone those without x) into the new ones, i.e.,

$$\begin{cases} \frac{5}{2}y - 2z + 5 \geq 5y - 2z + 7 \\ \frac{5}{2}y - 2z + 5 \geq \frac{2}{3}y + 2z - 4 \\ 2y - z + 3 \geq 5y - 2z + 7 \\ 2y - z + 3 \geq \frac{2}{3}y + 2z - 4 \\ -10y + z \geq -15 \end{cases}$$

In this way we eliminate x. We can repeat this process until we have only 1 variable left. In general we'll end up with inequalities of the following form:

$$\begin{cases} x_n \geq -3 \\ x_n \geq -6 \\ \cdots \\ x_n \leq 10 \\ x_n \leq -1 \\ \cdots \\ 0 \geq -2 \\ 0 \geq -10 \\ \cdots \end{cases}$$

Then the original LP is satisfiable if and only if the maximum of all q_i in inequalities $x_n \geq q_i$ is less than the maximum of all q_j in inequalities $x_n \leq q_j$, and every q_k in the inequalities $0 \geq q_k$ is non-positive. All inequalities derived are non-negative linear combinations of original constraints, so every q_i, q_j, q_k above are rational numbers.

Assume for now that the original constraints are satisfiable. Then we'll be able to pick a $x_n \in \mathbb{Q}$ that satisfies the inequalities we end up with. Substituting the x_n to the inequalities we get one step ago, we'll get some (satisfiable) inequalities of x_{n-1}, so we can keep back-substituting until we get a solution $x^* = (x_1, x_2, \cdots, x_n) \in \mathbb{Q}^n$. This proves that if an LP has solution it must also have a rational one.

We now assume that the original constraints are not solvable, so if we do another step of Fourier-Motzkin Elimination, we'll end up with $0 \geq c$ for some positive c, which is a non-negative linear combination of original constraints, thus proving the result. □

Notice that Fourier-Motzkin Elimination actually solves LP; however, it's not polynomial. During each step, if we start with k inequalities, in the worst case we may end up with $k^2/4 = \Theta(k^2)$ new inequalities. Since we start with m constraints and must perform n steps, in the worst case the algorithm may take $\Theta(m^{2^n})$ time, which is far too slow. We'll show how LP can be solved efficiently in the next section.

3 Equational form

Our next goal will be to solve LP in polynomial time. However, it is not immediately clear that LP is solvable in polynomial time; for example, if every solution of an LP requires exponentially-many bits to write down, then it would be impossible to solve LP in poly-time. In this section we'll prove that this will not be the case, i.e., if an LP is satisfiable, then we can always find a solution with size polynomial to input size.

Theorem 3.1. *Given input*

$$K = \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \end{cases}$$

with input size (number of bits needed to write the input) L, which we denote as $\langle K \rangle = L$,

(1) $K \neq \varnothing \implies \exists$ feasible solution $x^* \in \mathbb{Q}^n$ with $\langle x^* \rangle = poly(L)$;

(2) $K = \varnothing \implies \exists$ proof $\lambda_i's$ with $\langle \lambda_i \rangle = poly(L)$ for every $i \in [m]$.

Proof. Suffices to prove (1) (by Remark 2.3). Assume K is not empty. If we consider the geometric meanings of our constraints, we may see that the set of solutions is a closed convex set in $\mathbb{R}^n$. Here is an example in $\mathbb{R}^2$:

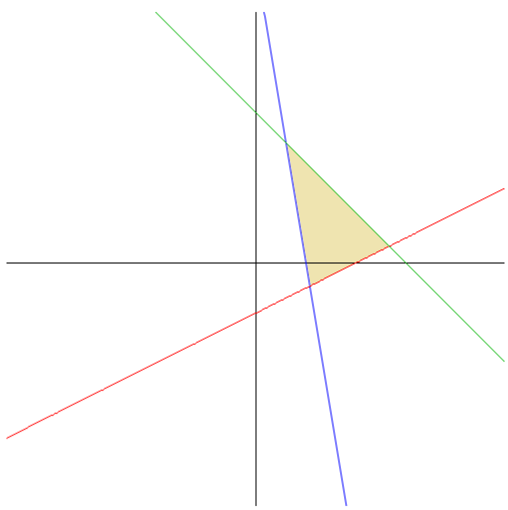

Figure 1: A Geometric view of LP

If $K \neq \varnothing$, then there exists some feasible vertices/extreme points/basic feasible solutions, which are the unique intersections of some linearly independent EQUATIONS out of the m EQUATIONS (we emphasize "equations" because constraints are originally given as inequalities, but we are now considering their equational form. Any vertex x^* of our solution set is a solution to a $n \times n$ system of equations, which can be found (and written down, of course) in polynomial time by Gaussian Elimination, and this proves that $\langle x^* \rangle = \text{poly}(L)$. $\square$

This proof basically sketches what we are trying to do, but it has problems; consider the following $n = 2, m = 1$ case:

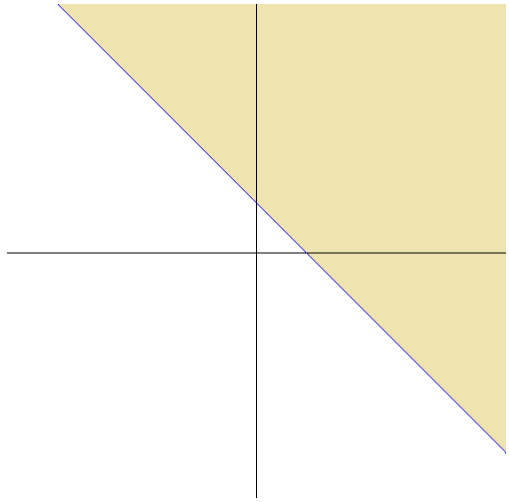

Figure 2: $x_1 + x_2 \geq 1$

We don't have any vertices in it! Then how can we find a polynomial-size solution? We observe, however, that we can solve the example above by adding the 2 axes. Before we move on, we give a definition:

Definition 3.2. We say K is included by a big box if $\forall i \in [n], x_i \leq B_i^+, x_i \geq B_i^-, \langle B_i^+ \rangle, \langle B_i^- \rangle =$ poly(L).

Observation 3.3. *The proof above would be fine if K is included by a big box.*

We now give another proof:

Proof. For each constraint $a^{(i)}x \geq b_i$, we replace it with $a^{(i)}x - S_i = b_i$ and $S_i \geq 0$, where the $S_i's$ are called slack variables. Then, replace each original x_i with $x_i^+ - x_i^-$ and $x_i^+ \geq 0$, $x_i^- \geq 0$. Then the new LP has $2n + m$ variables, each constrained to be non-negative, and all the other constraints are equations. We call this converted "Equational Form" of LP K'. Then K' has exactly the same solutions with K regarding the original variables, and $\langle K' \rangle = \text{poly}(\langle K \rangle)$. We write K' as $K' = \begin{cases} A'x' = b' \\ x' \geq 0 \end{cases}$, where A' is a $m' \times n'$ matrix, where $n' = 2n + m$, and $m' = m$. Assume WLOG that A' has rank m' (otherwise some constraints are unnecessary, so we can throw them out). Then K' is contained in a positive orthant of $\mathbb{R}^n$, and $A'x' = b'$ is an $(n' - m')$-dimensional subspace in $\mathbb{R}^n$. If $m' = n'$ then this subspace is a point, and this point x^* is a feasible solution with size poly(L), so we're done. Otherwise K is the intersection of this subspace and the positive orthant of $\mathbb{R}^n$. Since the subspace will not be parallel to all coordinates, it must intersect with some axes. Let the intersection be x^*. Then x^* is a solution to K' and $\langle x^* \rangle = \text{poly}(L)$. $\square$

To sum up what we have shown, given an LP problem that asks whether K is empty, we can, WLOG, include into K a big box, where each $\langle B_i^- \rangle, \langle B_i^- \rangle = \text{poly}(L)$, and if $K \neq \varnothing$, then we can always find a vertex x^* in K.

4 LP and reduction

Assume we already have a polynomial-time program that decides LP, i.e., it outputs "Yes" if $K \neq \varnothing$ and outputs "No" if $K = \varnothing$.

Q: How can we use this program as a black box to solve LP (i.e., outputs a point in $\mathbb{Q}^n$ in K if $K \neq \varnothing$)?

A: Suppose $K \neq \varnothing$. We only need to find $a_1, a_2, \cdots a_n \in \mathbb{Q}$ such that $K \cap \{\forall i \in [n] x_i = a_i\} \neq \varnothing$. How? From the last section we know that we can assume every variable x_i is bounded by B_i^- and B_i^+, where $\langle B_i^- \rangle = \text{poly}(L)$, $\langle B_i^+ \rangle = \text{poly}(L)$. Then for each x_i, we can do a binary search, starting with $B_i = (B_i^- + B_i^+)/2$. If our program tells us $K \cap \{B_i \leq x_i \leq B_i^+\} = \varnothing$, we know that $B_i^- \leq x \leq B_i$, so we continue to search whether $(B_i^- + B_i)/2 \leq x \leq B_i)$; otherwise we search whether $(B_i + B_i^+)/2 \leq x \leq B_i^+$, etc. The binary search takes at most $O(\log_2 B_i^+ - B_i^-) = O(\log_2 2^{\text{poly}(L)}) = \text{poly}(L)$ time for each variable, so we can efficiently find a point in K.

Q: What if K is something like $\{3x_1 = 1\}$? In that case the binary search may never end.

A: WLOG assume every $a_{ij} \in \mathbb{N}$ for $i \in [m], j \in [n]$ and $b_i \in \mathbb{N}$ for $i \in [n]$. Let $c = \prod |a_{ij}|$. Since each $a_i j$ has size L, i.e., $|a_{ij}| \leq 2^L$, we have $c \leq 2^{mnL} = 2^{\text{poly}(L)}$. Then if $K \neq \varnothing$, there must be some vertex v such that $cv_i \in \mathbb{Z}$ for every $i \in [n]$, and cv_i is bounded by cB_i^- and cB_i^+, both of which still has size $\text{poly}(L)$. Then we can do binary search efficiently.

Recall that another goal of LP is to maximize $c \cdot x$ for some $c \in \mathbb{Q}^n$ with the constraints K.

Q: Given a program that decides LP, How do we solve $\max\{c \cdot x : x \in k\}$?

A: We can add $\{c \cdot x \geq \beta\}$ as a constraint and do binary search to determine the largest β such that $K \cap \{c \cdot x \geq \beta\} \neq \varnothing\}$. We have to be a bit careful here, however, because the maximum can be infinity. To fix this, we have the following observation.

Observation 4.1. *Suppose c is not parallel to any a_i. If max$\{c \cdot x\}$ is not infinity, then the maximum must occur at a vertex.*

All vertices are bounded by B_i^- and B_i^+, so we can bound $\max\{c \cdot x\}$ with some M by choosing $B_i = B_i^+$ if $c_i > 0$ and $B_i = B_i^-$ if $c_i < 0$ and set $M = c \cdot B$. Now that the maximum is bounded, we can compute $\max\{c_i x_i | x \in K \cup \{c \cdot x \leq M + 1\}\}$ by the way we described above. If the maximum we get is $M + 1$, then it is actually infinity; otherwise we get the maximum we intended.

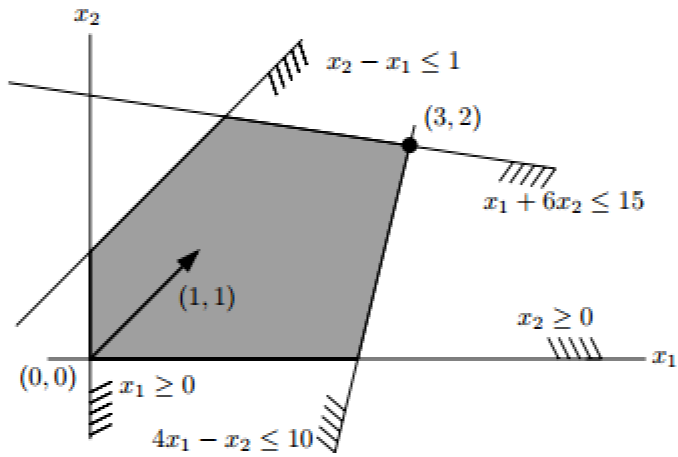

Figure 3: An example of maximizing $x_1 + x_2$

Now that we know how to maximize $c \cdot x$, we switch back to the first goal, and suppose $K \neq \varnothing$.

Q: Instead of any $x \in K$, how can we make sure we output a vertex?

A: From the observation, we may choose an arbitrary $c \in \mathbb{Q}^n$ that is not parallel to any constraints, and maximize $c \cdot x$. This guarantees to give us a vertex.

References

[Kha79] LG Khachiyan, *A polynomial-time linear programming algorithm*, Zh. Vychisl. Matem. Mat. Fiz **20** (1979), 51–68.

[MG06] Jirí Matouek and Bernd Gärtner, *Understanding and using linear programming (universitext)*, Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2006.