Lecture 14 - Linear Programming II
("Using Lin.Prog.")

[scribe]

A problem that's truly linear: Max-st-Flow.

Input: Digraph $G = (V,E)$

- "Capacity" $c_{uv} \in \mathbb{R}^+ \vee (u,v) \in E$
- "Source" $s \in V$
- "Sink" $t \in V$

You want to move "stuff" from $s \to t$.

First: Tolstoy USSR, 1930

\[ s = \text{tovary edges} = \text{railroads} \]

\[ s = \text{cement factory} \quad t = \text{needs cement} \]

\[ c_{uv} = \text{amt shippable/day (shortly: steel)} \]

How much can we produce/consume?

"Flow conservation": $\forall v \in V \setminus \{s,t\}$, incoming flow = outgoing flow.

\[ \text{To max flow: is a LP!} \quad \text{Wts: } f_{uv} \quad \forall (u,v) \in E \]

\[ \text{consts: } \quad f_{uv} \geq 0 \quad \forall (u,v) \in E \]

\[ f_{uv} \leq c_{uv} \quad \forall (u,v) \in E \]

\[ \forall v \in V \setminus \{s,t\}, \sum_{u: u \to v} f_{uv} = \sum_{w: v \to w} f_{vw} \]

"objective": $\max \sum_{v: s \to v} f_{sv} - \sum_{u: u \to s} f_{us}$.

That's it! Max-st-Flow. Also solvable by Ford-Fulkerson etc.

Aside: Solvers: Maple (full/real), Matlab/Sedumi, cplex... Yuck.

\[ \text{FF, 7am} \]

\[ \text{fact: all } c_{uv} \in \mathbb{Z} \]

\[ \text{every basic flow } z \text{ s.t. } \] is integral

\[ z \text{ is integer} \]
FF'SH: also used generic railroad example. Credited Ted Harris. Robert Corp. for the prob. In fact Harris was also working on the prob. of the Soviet railroad network.

What abt a problem that isn't purely linear?

Bipartite Perfect Matching We discussed RNC alg, but let's also see LP method. Also solvable using Max-Flow. But some mind.

Goal: Find max-wt. (bipartite) perfect match. All jobs must get done, want best utility.

[Could you have a left nut for... taking edges?]

Not on LP. But...? Very common paradigm: "Integer linear program" [ILP] "relax" Linear Program

vibs: \( \forall (u,v) \in E \quad x_{uv}, \text{ intent: } 1 \text{ if matched} \)
\[ \text{ILP} \quad \Rightarrow \quad \text{Relax} \quad \Rightarrow \quad \text{LP} \]

Constraints:
- \( \forall u \in U, \sum_{v \in V} x_{uv} = 1 \)
- \( \forall v \in V, \sum_{u \in U} x_{uv} = 1 \)
- \( \max \sum_{u \in U} w_{uv}x_{uv} \)

Relaxation facts:
- LP infeasible \( \Rightarrow \) ILP infeasible.
- LP feasible \( \Rightarrow \) either ILP infeasible or \( \text{Opt} \leq \text{LPOpt} \)

Why do this? Well, ILPs are NP-complete, but LPs in \( P \). What good is an opt. feasible soln. though?

Can think \( \square \) \( \text{LPOpt} = 83.6 \) is a proof that \( \text{Opt} = 83.6 \)
(Only time found)

Easy to compute u.b. on Opt.

Lucky (?) situation for Max-Lt Bip. P.M. Doesn't always happen.

That: All extreme pts are integral

\( \Rightarrow \) LP feasible means \( \text{LPOpt} = \text{Opt} \), problem \( \in P \)

Actually, our proof implicitly gives an effic. method converting any LP-Opt soln. to an optimal (hence integral) extreme pt.
IF \( \hat{x} \) feasible

\[ \text{Contrapositive:} \ \text{non-integral} \rightarrow \hat{x} \text{ not extreme} \]

\[ \hat{x} = \frac{1}{2} x^+ + \frac{1}{2} x^- \text{ for some fees} \]

\[ x^+, x^- \]

Must exist non-integer cycle.

\[ \exists \epsilon > 0 \text{ s.t.} \]

\[ 0 < \epsilon < 1 - \epsilon \text{ add edge on cycle} \]

Let \( x^+ \) be \( \hat{x} \) with \( \epsilon \) added
to odd edges of cycle,

\[ x^- \text{ be } \hat{x} \text{ even edges} \]

Both feasible, and (4) holds.

Fact: This "integrality" happens for any LP

\[ \max \text{ or } \min \ c^T x : \ 0 \leq Mx \leq b, \]

\[ 0 \leq x \leq u, \]

where \( b, b, I, u, u \subseteq \mathbb{Z} \cup \{\infty\}, \ M \text{ "totally unimodular"} \]

[Thank you!]

Every \( x \) s.t. \( c^T x \) has

\[ \det \geq -1, 0, 1 \]

\[ \text{(e.g., this polyhedron has \( \text{int} \) polytopes)} \]

\[ \text{Next we'll see an e.g., where the LP does not exactly solve the prob, but it's still good...} \]
Min-Vertex-Cover

Input: \( G = (V, E), \) \( u \) - \( v \) costs \( c_{uv} \geq 0 \)
Output: A \( \"\text{\text{\textit{vertex cover}}}\text{\text{\textit{s}}} \) \( S \subseteq V \) st. each \( u \in E \) has an end pt \( \in S \)
Goal: Min cost.

\[ \text{e.g.: } \begin{array}{c}
1 & \leftrightarrow & 2 \\
2 & \leftrightarrow & 3 \\
1 & \leftrightarrow & 3 \\
\end{array} \]
\[ \text{Opt} = 2 \]

Greedy = 6

\text{rem: NP-hard, so won't get integrally lucky.}

ILP:
\[ \begin{array}{c}
x_u \in \{0, 1\} \quad \forall v \in V \\
x_u + x_v = 1 \quad \forall (u, v) \in E \\
\min \sum_{v} c_{uv} x_u \\
\end{array} \]
\[ \text{LP: } 0 \leq x_v \leq 1 \]
\[ \text{LP opt} \leq \text{Opt} \]

\text{can be less: } \begin{array}{c}
1 & \leftrightarrow & 2 \\
2 & \leftrightarrow & 3 \\
1 & \leftrightarrow & 3 \\
\end{array}
\[ \text{LP opt} = 2, \quad \text{LP opt} \leq \frac{2}{3} \]
\[ G = K_n: \text{Opt} = n - 1, \quad \text{LP opt} \leq \frac{n}{2} \]

[What good is all this?]

Part of Opt 7:

**LP Rounding:** converting an optimal fractional (LP) sol to a (hopefully almost as good) ILP sol.

Say \( \tilde{x} \) feasible for LP. Think of it as optimal, too, but defines
\[ S = S_{\tilde{x}} = \{ v \in V: \tilde{x}_v \geq \frac{1}{2} \} \]
"rounding" \( \tilde{x} \) to a 0/1 sol.
**Fact:** $S$ is a vertex cover (ILP feasible).

**Prove:**

\[ \sum_{v \in S} \tilde{x}_v \geq \sum_{v \in S} \frac{1}{2} c_v = \frac{1}{2} \text{cost}(S) \]

**Cor:** Let \( x^* \) be opt. LP sol.

\[ \text{cost}(S_{x^*}) = 2 \text{LP cost}(x^*) = 2 \text{LP opt} \]

\[ \text{Opt} \leq \text{LP opt} \leq 2 \cdot \text{Opt} \]

Can effic. find sol. w/ factor 2 of Opt.

**Think:** For this LP, every \( v \times \tilde{x} \) is "half-integral".

\( \tilde{x}_v \in \{0, \frac{1}{2}, 1\} \)

**Duality:** Let's go back to max-st-flow.

\[
\begin{align*}
\text{max} & \quad c^\top x \\
\text{st.} & \quad a^i \cdot x \leq b_i, \quad \lambda_i \\
& \quad a^0 \cdot x \leq b_0, \quad \lambda_0 \\
& \quad \ldots \\
& \quad a^n \cdot x \leq b_n, \quad \lambda_n \\
& \quad = \quad c^\top x \leq \beta.
\end{align*}
\]

**Proof:** That \( \text{LP opt} \leq \beta \).

Farkas Lemma \( \exists \) optimal proof of this form.
I.e., \[
\begin{align*}
\min & \quad \mathbf{b} \cdot \mathbf{x} \\
\text{s.t.} & \quad A\mathbf{x} = \mathbf{c}
\end{align*}
\]
\(\leq \) an LP

\(\text{Dual} \) = \(\text{optimal LP's max.} \)

Life tip: Given any LP, always take its dual, try to "interpret" it.

\text{Max-Flow LP}

\[
\begin{align*}
\max & \quad \sum_{v:s \to v} f_{sv} - \sum_{u:s \to u} f_{us} \\
\text{s.t.} & \quad (\text{mess around a little})
\end{align*}
\]

\[
\begin{align*}
\text{"clean up"} & \quad \text{dual equivalent to} \\
\lambda_{uv} & \quad \text{vol} + \text{u} \in \mathcal{E} \\
\mu_v & \quad \text{vol} + \text{v} \in \mathcal{V}
\end{align*}
\]

"interp" \[
\begin{align*}
\min & \quad \sum_{u \in \mathcal{E}} C_{uv} \lambda_{uv} \\
\text{s.t.} & \quad \mu_s = 0 \\
& \quad \mu_t = 1 \\
& \quad \lambda_{uv} \geq 0 \quad \text{for} v \in \mathcal{E} \\
& \quad \lambda_{uv} \leq \mu_u - \mu_v
\end{align*}
\]

\text{Natural Relaxation of Min-st-Cut ILP:}

Find \(S \subseteq \mathcal{V}, S \neq \emptyset, S \neq \mathcal{V}\),

\[
\min \text{ cost}(S) = \sum_{u \in S} C_{uv}
\]

Duality:

\[
\begin{align*}
\text{max flow} & \quad \leq \text{min "fractional" s-t cut} \quad \leq \text{Min-st-cut}
\end{align*}
\]
Actually, min-st-cut LP is exact.

Max-Flow = Min-Cut.

Fact: If like for max-cut bipartite matching (Proof not hard).

In fact, LP is totally unimodal.

Shows the fact that if caps all integral, so are all 0's.

FF proved this (so did P. Elias-Fenstein-Shanon).

Ted Harris & Rand

Soviet rail network

Gen Ross & air force -> cared about min-cut for bombing purposes.

Q: Max-Cut ... ?