Lecture 6 - Special Graph Theory I [first 2/3]

[SCRIBE] [A topic beloved by alg/com people alike.]

[Technicalities] Finite

- multiple edges, self-loops: OK
- no deg-0 vertices  𝑖. 𝑖. 𝑖. 𝑖. isolated

Never seen this before? -> Assume G regular. Don't have to, but conceptually easier.

Everything we do generalizes to "reversible Markov chains." A lot of what... "non-reversible" "directed graphs".

We'll be looking at: Labeling V with real #s:

\[ f: V \rightarrow \mathbb{R} \]

e.g.: f is temperature, voltage, coord. of an embedding into \( \mathbb{R}^d \), 01-indic. of SC V...

\[ \sum_{v \in V} f(v) \]

rem: can add f+g (ptwise), scalar mult c⋅f \( \rightarrow \{ f: V \rightarrow \mathbb{R} \} \) a vector space.

[Form, they're vectors.]

Dimension? \( n \)

KEY TO SPEC. GR. THY:
$$E \left[ \left( f(u) - f(v) \right)^2 \right]$$

**Dirichlet form**

**Intuition:** “Small” $\iff$ $f$ “smooth”, doesn’t vary much on edges

**Key example:** $S \subseteq V$, $f = 1_S$ ($f(u) = \{1 \text{ if } u \in S\}$)

Then $E[f] = \frac{1}{2} \sum_{uv} E \left[ (1_S(u) - 1_S(v))^2 \right] = \frac{1}{2} \sum_{uv} E \left[ 1[(uv) \text{ "crosses" } S] \right]$

= $\frac{1}{2} \{ \text{frac. of edges on } \partial S \}$

Do you find $\frac{1}{2}$ annoying? It’s cool. Think of $uv$ directed. $f$ on $uv$, $\frac{1}{2}$ chance going “in”, $\frac{1}{2}$ chance "out"

\[= \text{Pr}_{uv} [u \to v \text{ is stepping out of } S] \]

This is something we care about—finding small “cuts” in graphs, finding “communities”, etc.
We're often choosing unif. rand edge \((u,v)\).  
We should probably talk abt. choosing rand vertex, too. 
Just unif random? No! Not "compatible" with our edge choice. 

To choose rand vts:  
1. choose edge \(uv\) unif.  
2. output \(u\) [or \(v\)].  

\(\text{(immed.) fact: } \Pi\text{ [v] is proportional to } \deg(v).\)  
\[ \Pi\text{ [v]} = \frac{\deg(v)}{|E|}. \]

\(\text{gives weight/importance to vts.}\)

\(\text{if } G\text{ regular, } \Pi\text{ is uniform dist. } \) 
(\text{This is why it's easiest to think of regular case.})

\(\text{fact: } \{\hspace{1em} \text{draw } u \sim \Pi \hspace{1em} \} \quad \{\hspace{1em} \text{let } v \text{ be unif. rand} \hspace{1em} \} \equiv \text{ draw unif rand edge } uvv.\)

Why? [condit: on } u, \text{ equally likely to have came } \{v, \text{ any nbr}\} \text{ or, look at formula for } \Pi, \text{ mult by } \deg(u).\)

\(\Rightarrow \text{ distib. of } v \text{ is also } \Pi.\)

Cor:  
Let } t \in N. \quad [\text{e.g., } t = 17]  
Draw } u \sim \Pi.  
Do rand. walk for } t \text{ steps: } u \overset{t}{\longrightarrow} v.  
Then } v \text{ has distib. } \Pi.  

term: } \Pi \text{ is invarnt/stat. distrib.}
Q: Say $u_0 \in V$ not random. [Worst case II]
Do $u_0 \xrightarrow{\ell} V$. [Random variable II]

As $\ell \to \infty$, does distribution of $v \to \pi$?

A: Not if ... $G$ disconnected,
   
   - $G$ bipartite
   
   Otherwise yes! [We'll see why later,
   so it's also "limiting distr." except in some "freak cases,"

Q: How fast?
A: [We'll see... I spectral [eigenvalue] considerations... e.g.

\[ E[1_S] "small". [We'll see... II
Fast converge $\iff E[f] "never"
small" "asymptotically"

"Small"? Compared to what? $E[e^f] = e^{E[f]}$, so we have a "scaling issue", Can just make $f$ have huge values. Like variance, should normalize to $1$...]

\[ E[f(u)] = E[f] [\text{short hand}]. \text{If } S \cap V, E[1_S] = \Pr [u \in S]

\[ \text{Var } f(u) = E_i \left[ (f(u) - \mu)^2 \right] = E_i \left[ f(u)^2 \right] - \left( \frac{\mu}{1} \right)^2 \]

\[ = \frac{1}{N} \sum_{u \in V} \left( f(u) - f(v) \right)^2 \text{ ind. dep.} \]

\[ \text{this is just totally normal fact abt RVs: } \text{Var } x = \frac{1}{N} E[(x-x_N)^2] \]
\[
\begin{align*}
\mathbb{E}\left[ f \right] &= \frac{1}{\mathcal{V}(G)} \mathbb{E}_{u \in \mathcal{V}(G)} \left[ f(u) \right] \\
\mathbb{V}(f) &= \frac{1}{\mathcal{V}(G)} \mathbb{E}_{u \in \mathcal{V}(G)} \left[ f(u) - \mathbb{E}[f(u)] \right]^2
\end{align*}
\]

"local var. (or edges)"

"global var."

[SEPARATE BOARD]

**Definition:** Let \( f, g : V \to \mathbb{R} \).

\[ \langle f, g \rangle_{\mathbb{P}} := \mathbb{E}_{u \in V} \left[ f(u) g(u) \right] \]

"correlation" of \( f, g \);

how "similar" they are.

\[ \langle f, f \rangle_{\mathbb{P}} \] is normal "dot prod", scaled by \( \frac{1}{\mathcal{V}(G)} \),

if \( G \) regular. \( \| f \|_{\mathbb{P}} \) not true, and this screws people up.

\[ \| f \|_{\mathbb{P}}^2 = \mathbb{E}_{u \in V} \left[ f(u)^2 \right] \geq 0 \]

equality \( \iff f = 0 \).

**Key remark:** Let \( S \subseteq V \), \( f = 1_S \).

\[ \| f \|_1 := \mathbb{E}_{u \in V} \left[ f(u) \right] = \mathbb{P}(u \in S) = \text{"vol}(S)". \]

\[ \| f \|_2^2 := \mathbb{E}_{u \in V} \left[ f(u)^2 \right] = \mathbb{E}[f(u)] = \frac{1}{\mathcal{V}} \sum_{\text{edges}} \mathbb{E}[f(u)]^2 \]

\[ \langle f, f \rangle := f \text{ \# 1-valued.} \]
(Back to (c) (We said fast cusp of rand walks $\iff \mathbb{E}[f]$ "never small."

How small can it be?

Certainly $> 0$. Can it actually be 0? Sure, if $f=0$.

$\mathbb{E}[f]=0$ if $f=0$, but is there nontrivial such $f$?

Yes, $f=1$. $\mathbb{E}[1]=0$ for any const. $\mathbb{E}$.

Non-const. sol's?

(Say we try to get $\mathbb{E}[f]=0$.)

Prop: $\mathbb{E}[f]=0$ iff

$f$ is const. on each connected comp. of $G$.

& # conn comps $= \# \text{ lin. indep. } f \text{ s.t. } \mathbb{E}[f]=0$.

$\implies$ If c.c.'s are $S_1, \ldots, S_k$, $1_{S_1}, \ldots, 1_{S_k}$ are lin indep.

And in general, all $f$ s.t. $\mathbb{E}[f]=0$ are $\frac{1}{k} c_i 1_{S_i}$.

Key idea of spectral gh theory is "robustification" of this:

$G$ has $k$ "mostly disconnected" comps. iff $\exists k$ lin indep $f$ with $\mathbb{E}[f]$ "small". "Sparset Cut" apps.

Maximizing $\mathbb{E}[f]$?

[Need scaling const., $: \mathbb{E}[cf]=c^2 \mathbb{E}[f]$]

(Natural choice, given $\mathbb{E}$). Max $\mathbb{E}[f]$ s.t. $\text{Var}[f]=1$

$\leq$ some, \; i f not 1, m ult by const $= 1$.

\\[
\text{or sometimes we consider } f \text{ s.t. } ||f||_2^2 = \mathbb{E}[f^2] \leq 1.
\]

Same thing! Why? $I = \min \mathbb{E}[f] \text{ s.t. } \mathbb{E}[f] \geq 1$, $\mathbb{E}[f] = \mathbb{E}[f]$

Choose $\alpha$ s.t. $\alpha \mathbb{E}[f] \geq \alpha$, and becomes $\text{Var}[f]$. 

\(\text{Remark:}\)
Max $E[f]$? Intuition: embed $V \rightarrow \mathbb{R}$ line so edges "far apart."

Q: for what kind of $G$ can you be most successful?

A: bipartite?

\[ \max E[f] \text{ s.t. } E[f^2] \leq 1 \]

If $G$ bipartite, $V = (V_1, V_2)$, let $f = 1_{V_1} - 1_{V_2}$:

\[ E[f^2] = 1 \quad E[f] = 2 \quad \frac{1}{2} \sum_{u \in V} E[(f(u) - f(v))^2] \]

[least possible?]

prep: $E[f] \leq 2 \| f \|_2^2$ always

pf: $\frac{1}{2} \sum_{u \in V} E[(f(u) - f(v))^2] = \frac{1}{2} \sum_{u \in V} E[f(u)^2] + \frac{1}{2} \sum_{u \in V} E[f(v)^2] - 2 \sum_{u \in V} E[f(u)f(v)] \overset{\text{in abs. vol.}}{\leq} \frac{1}{2} \| f \|_2^2$ in abs. vol.

\[ \frac{1}{2} \| f \|_2^2 \leq \frac{1}{2} \sum_{u \in V} E[f(u)] E[f(v)] \]

\[ = \frac{1}{2} \| f \|_2^2 \]

possible $\iff$ G bip.

Eq. almost possible iff $G$ "close to" bipartite?

\[ \text{Max-Cut approx alg.s.} \]

Enter eigenvalues...