Lecture 4.5:
Discriminating Two Qubits
Discriminating Quantum States:

Given an unknown quantum state $|\psi\rangle$.

You’re promised it’s either $|u\rangle$ or $|v\rangle$.

(These are two states you know.)

Must guess whether $|\psi\rangle = |u\rangle$ or $|\psi\rangle = |v\rangle$. 

Input qubit at angle $\epsilon$

Dud: Qubit at angle $\epsilon$

Bomb: Qubit at angle 0 (assuming no explosion)
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**Discriminating Quantum States:**

Measure in some basis?

Measure in the **standard basis**...
Measure in some basis?

Measure in the **standard basis**...
- If readout $|0\rangle$, guess $|v\rangle$
- If readout $|1\rangle$, guess $|u\rangle$

**Error?**
- If $|\psi\rangle = |u\rangle$ then $\Pr[\text{error}] = (\cos \gamma)^2$, where $\gamma = \text{angle between } |u\rangle$ and $|0\rangle$
  $$= \cos^2(45^\circ + \theta / 2)$$
  $$= \frac{1}{2} - \frac{1}{2} \sin \theta$$
Measure in some basis?

Measure in the **standard basis**...

- If readout $|0\rangle$, guess $|v\rangle$
- If readout $|1\rangle$, guess $|u\rangle$

Error?

- If $|\psi\rangle = |u\rangle$ then $\Pr[\text{error}] = \frac{1}{2} - \frac{1}{2} \sin \theta$
Measure in some basis?

Measure in the **standard basis**...

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Error?

- If $|\psi\rangle = |u\rangle$ then $\Pr[\text{error}] = \frac{1}{2} - \frac{1}{2} \sin \theta$
- If $|\psi\rangle = |v\rangle$ then $\Pr[\text{error}] = \frac{1}{2} - \frac{1}{2} \sin \theta$
Measure in some basis?

Measure in the **standard basis**...

- If readout $|0\rangle$, guess $|v\rangle$
- If readout $|1\rangle$, guess $|u\rangle$

**Error?**

- If $|\psi\rangle = |u\rangle$ then $\Pr[\text{error}] = \frac{1}{2} - \frac{1}{2} \sin \theta$
- If $|\psi\rangle = |v\rangle$ then $\Pr[\text{error}] = \frac{1}{2} - \frac{1}{2} \sin \theta$

This is a “two-sided error” algorithm.
A “one-sided error” algorithm?

Measure in the \( \{|u\rangle, |u\perp\rangle\} \) basis...

- If readout \(|u\rangle\), guess \(|u\rangle\)
- If readout \(|u\perp\rangle\), guess \(|v\rangle\)

Error?

- If \(|\psi\rangle = |u\rangle\) then \( \Pr[\text{error}] = 0 \)
- If \(|\psi\rangle = |v\rangle\) then \( \Pr[\text{error}] = (\cos \theta)^2 \)

\[ = 1 - (\sin \theta)^2 \]

“No false positives”
(\( |v\rangle \) = bomb = ‘positive’)

\( |u\rangle \) and \( |u\perp\rangle \) are orthogonal.

\( \theta \) is the angle between \( |u\rangle \) and \( |v\rangle \).

\( \langle v | u \rangle = 1 \)

same as prob. of explosion
A **one-sided error** algorithm?

Measure in the \{\ket{u}, \ket{u^\perp}\} basis...

- If readout \ket{u}, guess \ket{u}
- If readout \ket{u^\perp}, guess \ket{v}

Error?

- If \ket{\psi} = \ket{u} then \text{Pr}[\text{error}] = 0
- If \ket{\psi} = \ket{v} then \text{Pr}[\text{error}] = (\cos \theta)^2

\[= 1 - (\sin \theta)^2\]
A “zero-sided error” algorithm? We have a “no false positives” algorithm.

By symmetry, we have an equally good “no false negatives” algorithm.

For a zero-sided error algorithm:

• With probability $\frac{1}{2}$, do no-false-positives test;
  With probability $\frac{1}{2}$, do no-false-negatives test
  • If you get the answer you’re “sure of”, guess it;
    Otherwise, output “don’t know”

\[
\Pr[\text{don’t know}] = \frac{1}{2} + \frac{1}{2} \Pr[\text{error in one-sided alg.}] = 1 - \frac{(\sin \theta)^2}{2}
\]
A “zero-sided error” algorithm?

We have a “no false positives” algorithm.

By symmetry, we have an equally good “no false negatives” algorithm.

\[
\text{Pr[don't know]} = \frac{1}{2} + \frac{1}{2} \text{Pr[error in one-sided alg.]} = 1 - \frac{(\sin \theta)^2}{2}
\]
Discriminating Two Quantum States at Angle $\theta$

Two-sided error: $\frac{1}{2} - \frac{1}{2} \sin \theta$

One-sided error: $1 - (\sin \theta)^2$

Zero-sided error: $1 - \frac{(\sin \theta)^2}{2}$

Graph showing error probability as a function of $\theta$.

Diagram showing quantum states $|u\rangle$ and $|v\rangle$ at an angle $\theta$.
Discriminating Two Quantum States at Angle $\theta$

Two-sided error: $\frac{1}{2} - \frac{1}{2} \sin \theta$

One-sided error: $1 - (\sin \theta)^2$

Zero-sided error: $1 - \frac{(\sin \theta)^2}{2}$