Lecture 3: Understanding & Measuring 1 qubit

(in computing we have concept of...) (implemented on a
physical computer/chip)

logical bit

physical bit

0

low voltage

1

high voltage

(e.g. or magnetization
not)

(Maybe using 1 million electrons to store 1 bit.
We're into miniaturization. Could we make a bit
out of 1 particle, like an electron or photon?)

(Understanding properties of such subatomic particles is
the domain of Q. M.)

(Need a particle "property" with 2 possibilities,
which we can then call 0/1.)

(hydrogen atom has 1 electron, say it
can be in 2 levels)

electron "spin"
up: "0"
down: "1"

(I don't really know
what this "is"
how to measure
using magnetism.)

photon polarization
horizontal ↔ "0"
vertical ↑ "1"

(You've heard of this.
Pol. filters for cameras,
polarizing sunglasses,
3-D glasses at movies)
(Take photons & polarization, since kinda familiar. Take my word for it, one can build this machine...)

\begin{equation}
\text{measuring device}
\end{equation}

\begin{equation}
\text{Horz}\leftarrow \text{Digital readout which says "Horz" or "Vert" depending on if photon's polarization measured to be } \leftrightarrow \text{ or } \uparrow \right)
\end{equation}

(\text{Traditional diagram: readout via a needle pointing})

(Great, so use Horz=0, Vert=1, start guiding computer? Not so fast...)

\textbf{Q.M. Law #1:} If a "quantum system"/"particle" can be in \text{of} \text{of} \text{two basic states} \text{(0\rangle or (1\rangle)}, it can also \text{be in a superposition state}, meaning:

\begin{equation}
\alpha \text{ "amplitude" on } 0\rangle, \\
\beta \text{ "amplitude" on } 1\rangle,
\end{equation}

where \(\alpha, \beta\) are \text{numbers satisfying} \(|\alpha|^2 + |\beta|^2 = 1\).

("amplitude": just some word) "Qubit"
E.g., a photon may have state

\[ 0.8 \text{ amplitude on } |0\rangle, \quad (\text{horz. polarization}) \]
\[ 0.6 \text{ amplitude on } |1\rangle, \quad (\text{vert.} \quad \cdot \quad ) \]

(check: \(0.8^2 + 0.6^2 = 0.64 + 0.36 = 1\) ✓)

\[ \text{OR} \]

\[ 0.8 \text{ ampl. on } |0\rangle, \quad -0.6 \text{ ampl. on } |1\rangle \]

\[ \text{OR} \]

\[ 1 \text{ amplitude on } |0\rangle, \quad 0 \text{ amplitude on } |1\rangle \]

\[ \text{OR} \]

\[ i \text{ amplitude on } |0\rangle, \quad 0 \text{ amplitude on } |1\rangle \]

Yes, amplitudes are complex numbers!

(check: \(|i|^2 + |0|^2 = 1 + 0 = 1\) ✓

(However, because we’re already introducing a bit of wackiness, I’ll mostly stick to possibly negative real numbers in all my examples.)
(You might think "wait, you told me that measuring device for polarization exists... and it never reads out "MIXED", just "HORIZ" or "VERT". So what's the deal?)

Q.M. Law #2: For particle with amplitude $\alpha$ on $|0\rangle$, 

$\beta$ on $|1\rangle$,

if you measure it then...

with probability $|\alpha|^2$, readout shows "$|0\rangle$"

with probability $|\beta|^2$, "$|1\rangle$"

$= 1 \checkmark$ (makes sense)

(Deep & mysterious question: what exactly constitutes a "measurement". For our purposes, as the quote goes, physicists "know it when they see it".)

(BTW, in the prev. lecture, we were calling these two amplitudes "$f(0)"$ and "$f(1)"."

And: if readout is "$|0\rangle$", particle's state changes to "$1$ ampl. on $|0\rangle$,

0 "" "$|1\rangle$",

if readout is "$|1\rangle$", 

"0 ampl. on $|0\rangle$",

1 "" "$|1\rangle$. 

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E.g.: (this is a typical "quantum circuit diagram")

64% chance of "HORZ" |0⟩ 100% chance
36% ↓ "VERT" |1⟩

.init. state
0.8 on |0⟩
0.6 on |1⟩

1 on |0⟩
0 on |1⟩

HORZ (for example)

Same if init. state is "0.8 on |0⟩, -0.6 on |1⟩".

(But this is a fundamentally different state.
As we'll see, there are physical devices
that have different detection behaviors
for these two states.)
(Laws 1 & 2 are true for any particle property that has 2 basic states. For polarization, doesn't seem so bad. For, say, "position", can seem very weird.)

(forn simplicity, say retina has 2 sensitive positions; can call them 0 & 1)

\[
\begin{align*}
\text{1 photon source} & \quad \text{screen with 2 slits} \\
\text{(photon's position state: } & \frac{1}{\sqrt{2}} \text{ or } 1) \\
\text{frog retina} & \quad \text{(debated if human retina can detect 1 photon; frog retina definitely can)} \\
\end{align*}
\]

(Weird. Frog retina is the measuring device of Q.M. Law 2. Photon goes thru... both slits?
Well... that's how it is. You can do the physical experiment.)

(Similar story for photon thru a thin slice of glass... state becomes some ampl. on reflection, some ampl. on transmission!)
(Can actually illustrate using polarizing filters. I hesitate a bit because you have to think a bit carefully when thinking about them as meas devices)

"horz. filter"

0 measures photon pol. state \(\rightarrow |0\rangle \text{ or } |1\rangle\)

\[\text{if now } |0\rangle \leftrightarrow \text{ photon flies thru}\]

\[\text{else if now } |1\rangle \text{: photon converted to heat}\]

(If you fire laser pointer at it, can consider the millions of photons to be "randomly" polarized. Then 50% fly through. (It's a filter! Used in photography.) And those that do are all horz. polarized, Useful: it's like we now have a bunch of photons "initialized" to \(|0\rangle\).)

(Also exists a "vertical filter", where \(\oplus\) is reversed. What's cute: you can obtain it by physically rotating horz filter 90°, 3-d glasses have one for each eye.)

(What if you rotate 45°? We'll see...!)
Particle with 3 basic states

(Perfectly possible. 3 energy levels, or "spin" of a 3 positions. "deuterium nucleus."

| 1⟩, | 2⟩, | 3⟩ : a "qutrit"

With 4 basic states .... a "qudit" with dimension d = 4.

Most commonly: joint basic state of 2 qubits

e.g. 2 photons:

| ↔, ↔ | 1⟩ | 0⟩
| ↔, ↑ | 2⟩ | 1⟩
| ↓, ↔ | 3⟩ | 1⟩
| ↓, ↑ | 4⟩ | 1⟩

(We use the math convention of 1, 2, 3,..., d.
Might have been more elegant to use the python convention of 0, 1, 2,..., d-1,
because for d=2 we always use 0 & 1. Oh well.)

Q.M. Law 1: general state is

"amplitude α₁ on |1⟩, α₂ on |2⟩,
...
αₙ on |d⟩, such that |
|α₁|^² + ... + |αₙ|^² = 1"

(As last lecture's notation: N-dim. state, amplitudes f(0),..., f(N-1))

Q.M. Law 2: measurement: readout "|i⟩" with prob. |αᵢ|^², and then state becomes 1 ampl. on |i⟩, 0 on rest.
(Time to begin the math properly!)
(For a qudit we need to track a list of \( d \) numbers whose squared magnitudes sum to 1. That’s nothing more than…)

\[
\text{qudit state } \equiv \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{bmatrix} \in \mathbb{C}^d
\]

\( d \)-dim. column vector \( \rightarrow \vec{v} \)  

(Nontraditional notation in Q.M. We’ll change it soon.)

\[ |\alpha_1|^2 + \ldots + |\alpha_d|^2 = \| \vec{v} \|^2 = 1 : \text{ a unit vector} \]

\[ \text{e.g.: qubit state } \quad \frac{.8 \text{ amplitude on } |0\rangle,}{-0.6 \text{ amplitude on } |1\rangle} \]

\[ = \begin{bmatrix} .8 \\ -.6 \end{bmatrix} \]

(all blue points, unit circle correspond to qubit states)
(Actually, that picture is only appropriate for real amplitudes :) )

Recall: for complex $z = x + iy$,

$$|z|^2 = x^2 + y^2 = (x + iy)(x - iy) = z \cdot z^*$$

aka $\overline{z}$

(complex conjugate)

(squared magnitude of complex vector is sum of squares of all real & imaginary parts)

CONFUSING: One qubit $\to$ 2 (complex) numbers

One complex # $\to$ 2 real numbers

(we like to draw both of these in the 2-d plane)

(technically bogus for qubits: they need 4 real #’s)

(But we love to draw qubits in the plane so much, we’ll almost always do so. We’ll mainly only concern ourselves with real amplitudes. (till Shor’s Alg.) and we’ll try not to illustrate 1 complex # in plane.)
Unit vector: \( \| \tilde{v} \| ^2 = 1 \)

where \( \langle \tilde{u}, \tilde{v} \rangle \) is inner (dot) product

Meanings of \( \langle \tilde{u}, \tilde{v} \rangle \), \( \tilde{u}, \tilde{v} \in \mathbb{R}^d \)

1. (len. of \( \tilde{u} \)) \cdot (len of \( \tilde{v} \)’s projection on \( \tilde{u} \))
   (best way to think of it; 0 if \( \tilde{u} \) and \( \tilde{v} \) are perp., (length)\(^2\) if \( \tilde{u} = \tilde{v} \))

2. \( \| \tilde{u} \| \cdot \| \tilde{v} \| \cdot \cos \theta \). Just \( \cos \theta \) if \( \tilde{u}, \tilde{v} \) are unit (e.g., quantum states)

3. \( \tilde{u}_1 v_1 + \tilde{u}_2 v_2 + \ldots + \tilde{u}_d v_d = [\tilde{u}_1, \ldots, \tilde{u}_d] \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} = \tilde{u}^\dagger \tilde{v} \) (sum of squares of coords when \( \tilde{u} = \tilde{v} \))

Complex case: (We really want \( \langle \tilde{u}, \tilde{v} \rangle = \| \tilde{v} \| ^2 = \sum |v_i|^2 \) to hold.)

\[
\langle \tilde{u}, \tilde{v} \rangle \overset{\text{def}}{=} \tilde{u}_1^* v_1 + \tilde{u}_2^* v_2 + \ldots + \tilde{u}_d^* v_d
\]

\[
= \begin{bmatrix} \tilde{u}_1^* & \tilde{u}_2^* & \ldots & \tilde{u}_d^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix}
\]

\[ \tilde{u}^\dagger \] (\( \tilde{u} \)-dagger, conjugate transpose) (these formulas still ok in the real case)
Dirac's Bra-Ket notation

High school: \( \hat{1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( \hat{2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( \hat{3} = \begin{bmatrix} \frac{3}{2} \\ - \frac{\sqrt{2}}{2} \end{bmatrix} \) = \( 3\hat{1} + 5\hat{2} - 2\hat{3} \)

College: \( e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) ... \( e_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

Quantum: \( \begin{bmatrix} \vert 1 \rangle \\ \vert 2 \rangle \cdots \vert d \rangle \end{bmatrix} \)

(I love this notation. Used to hate it! I use it wherever \( d \) do linear, even if no quantum!)

Exception: \( d = 2 \): \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vert 0 \rangle \), \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vert 1 \rangle \)

2-qubit state \( \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \vert 0 \rangle + \beta \vert 1 \rangle \)

(Advantage 1: often deal with sparse vectors.
Advantage 2: no need to cram \# into subscript.)
notation: $|\text{Blah}\rangle$

$\uparrow$ any name

$|\cdot\rangle$ signifies type = column vector

Called a "ket".

notation: $\langle\text{Blah}|$ denotes its conjugate transpose, $|\text{Blah}\rangle^\dagger$, a row vector.

Called a "bra". (Bra-ket = bracket. Haha? Thanks, Dirac.)

notation: $\langle \bar{u}, \bar{v} \rangle$ is $u^\dagger v$

$= \langle u| \cdot |v\rangle$

$= "\langle u|v\rangle"$

(genuinely convenient notational shorthand)
example qubit: \[ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \] (so famous/important, has its own name:)

\[ = "|+\rangle" \]

Also: \[ |\rightarrow\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \]

\[ |+\rangle \rightarrow \frac{1}{\sqrt{2}} \overset{\text{w.prob } \frac{1}{2}}{\rightarrow} |0\rangle \quad \text{"|0\rangle" w.prob } \frac{1}{2} \]

\[ |1\rangle \overset{\text{with prob } \frac{1}{2}}{\rightarrow} \]

(Same if you feed in \(|\rightarrow\rangle\). Though these are different states.)

\[ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \cos\theta |0\rangle + \sin\theta |1\rangle \]

\[ \text{Pr[measuring } |0\rangle \text{]} \]

\[ \text{Pr[meas. } |0\rangle \text{]} = |\langle 0|\psi\rangle|^2 \approx (\cos\theta)^2 \]

(Important formula)

\[ |\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \]

\[ \langle 0|\psi\rangle \quad \langle 1|\psi\rangle \]

\[ \text{(len of } |\psi\rangle \text{ projected on } |0\rangle = |1\rangle) \]

\[ \text{Pr[meas. } |0\rangle \text{]} = |\langle 0|\psi\rangle|^2 \]
Measuring in a different basis

(Say you have a qubit in this state)

"standard" measuring device

- readout "|0\rangle" with prob. $|\langle 0|\psi\rangle|^2 = (\cos \theta)^2$
- readout "|1\rangle" with prob. $|\langle 1|\psi\rangle|^2 = (\sin \theta)^2$

(In fact...)

For any orthonormal basis $\{|u\rangle, |v\rangle\}$, can build a "measuring device for this basis".

$|\psi\rangle = \sqrt{\alpha}|u\rangle + \sqrt{\beta}|v\rangle$:
- readout is "|u\rangle" with prob. $|\langle u|\psi\rangle|^2$,
  i.e., $\cos^2(\text{angle between } |u\rangle, |\psi\rangle)$
- "$|v\rangle" with $|\langle v|\psi\rangle|^2$,
  i.e., $\sin^2(\text{angle between } |v\rangle, |\psi\rangle)$

E.g.: Measuring in the $\frac{1}{\sqrt{2}}|\uparrow\rangle, \frac{1}{\sqrt{2}}|\downarrow\rangle$ basis.
- $|\uparrow\rangle$: always reads out "|\uparrow\rangle"
- $|\downarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$: reads out "|\uparrow\rangle" w.p. $\frac{1}{2}$, "|\downarrow\rangle" w.p. $\frac{1}{2}$.

Example:

$|\psi\rangle = \frac{1}{\sqrt{2}}(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle) + \frac{1}{\sqrt{2}}(\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle)$

We love to do this. If your state is $f(0)|0\rangle + f(1)|1\rangle$, then it's $\frac{1}{\sqrt{n}} f(0)|\uparrow\rangle + \frac{1}{\sqrt{n}} f(1)|\downarrow\rangle$ and measuring in $|\uparrow\rangle, |\downarrow\rangle$ basis is like sampling from $f$'s Fourier coefficients...
Fun: can build a $\frac{1}{2}, 1\rightarrow \frac{1}{2}$ polarizing filter by physically rotating a horizontal one $45^\circ$.

(Do actual experiment...) (all passing photons now Horiz = 10)

\[
\begin{array}{c}
\text{laser} \quad \text{100%} \\
\text{brightness} \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{filter} \\
\text{50%} \\
\text{brightness} \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{screen} \\
\text{0%} \\
\text{brightness} \\
\end{array}
\]

(Recall: measures in Std basis.
Then: if 10) \rightarrow \text{heat}
if 11) \rightarrow \text{pass thru.}

Now interpose \text{filter} ($\frac{1}{2}, 1\rightarrow \frac{1}{2}$ measurer) at \text{*}.

(Childlike intuition: Currently no light at the end. Surely after another impediment, still no light.)

\[
\begin{array}{c}
\text{10} \\
\text{with prob } \frac{1}{2}, \text{ measures } 1+, \text{ passes thru } \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{1+} \\
\text{with prob } \frac{1}{2}, \text{ 10 measured: heat} \\
\end{array}
\]

So 25% brightness now!

12.5% brightness at end!!

with prob $\frac{1}{2}$, 11) measured: passes thru.
(A weak example of so-called "Quantum Anti-Zeno Effect," explored on Homework 2, Related to the "Elitzur--Vaidman Bomb," which we'll start with on Thursday!)