Lower bound for $K_G$ (Based on ideas of U. Haagerup).
Let $S$ denote the unit sphere in $\mathbb{R}^n$ or $\mathbb{C}^n$, let $\sigma$ denote the normalised area measure of $S$, and let $B$ denote the set of all measurable (real or complex, as appropriate) functions $f$ on $S$, with $|f(x)| \leq 1$ for all $x \in S$. For $0 < \rho < 1$, define

$$M_\rho = \sup_{f \in B} \left| \int_S f(y)g(x) < x,y > d\sigma(x)d\sigma(y) - \rho \int f(x)g(x)d\sigma(x) \right|$$

Then

$$K_GM_\rho \geq \rho \int_S f(y)g(x) < x,y > d\sigma(x)d\sigma(y) - \rho \int f(x)d\sigma(x) = 1 - \rho$$

So

$$K_G \geq \sup_{0 < \rho < 1} \frac{1 - \rho}{M_\rho}$$

We obtain an upper bound for $M_\rho$. Note that

$$M_\rho = \sup_{f \in B} \left| \int_S f(y)g(x) < x,y > d\sigma(x)d\sigma(y) - \rho \int fgdx \right|$$

Fix $f,g \in B$. Let $h(x) = f < x,y > f(y)d\sigma(y)$. Then $h$ is linear and so there is $z \in S$ and a scalar $\lambda$ so that $h(x) = \lambda < x,z >$ for all vectors $x$. We have $\lambda = h(z) = f < z,y > f(y)d\sigma(y)$. Let us write $\mu = f < z,x > \overline{g}(x)dx$. Then

$$\Re \left\{ \int_S f(x)g(x) < x,y > d\sigma(x)d\sigma(y) \right\} = \Re \mu = |\lambda + \mu|^2/4$$

and since $\lambda + \mu = f(f(x) + \overline{g}(x)) < x,z > d\sigma(x)$ it follows that

$$M_\rho \leq \sup_{f \in B} n \left[ \int_S (f(x) + \overline{g}(x))z_1d\sigma(x) \right]^2 - \rho \Re \int fgdx$$

This inequality holds in both real and complex cases (in the real case, the $\Re$ and complex conjugate symbols can of course be omitted). From now on, we consider the two cases separately.

**Real case.** In this case we have $f(x) + g(x) \leq 1 + f(x)g(x)$ whenever $f,g \in B$, so, writing $\phi(x) = (1 + f(x)g(x))/2$, we have

$$M_\rho \leq \sup_{f \in B} n \left\{ \int (\phi(x)x_1d\sigma(x))^2 + \rho(1 - 2\phi(x)) \right\}$$

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where $T$ is the set of measurable $\phi$ with $0 \leq \phi \leq 1$. For a given value of $f \phi dr$, the expression on the right will be maximized by choosing $\phi = 1$ when $|x_1| > \lambda$ and 0 otherwise, for appropriate $\lambda$. Hence

$$M_\rho \leq \sup_{0 < \lambda < \infty} n \left( \int_{|x_1| > \lambda} |x_1| |dr| \right)^2 + \rho \left( 1 - 2\sigma \{ |x_1| > \lambda \} \right)^2 + \rho \left( 1 - 2\sigma \{ |x_1| > \lambda \} \right)^2.$$

Now let $n \to \infty$; then the distribution of $n^{1/2} x_1$ under $\sigma$ tends to standard normal, and we have

$$M_\rho \leq \sup_{0 < \lambda < \infty} \left( \int_{\lambda}^{\infty} 2\sqrt{2\pi e^{-z^2/2}} dz \right)^2 + \rho \left( 1 - 2\sqrt{2/\pi} \int_{\lambda}^{\infty} e^{-z^2/2} dz \right)^2.$$

or $M_\rho \leq \sup_{0 < \lambda < \infty} F_\rho(\lambda)$ where

$$F_\rho(\lambda) = \frac{2}{\pi} e^{-\lambda^2} + \rho \left( 1 - 2\sqrt{2/\pi} \int_{\lambda}^{\infty} e^{-z^2/2} dz \right)$$

Consideration of the derivative of $F_\rho$ shows that, if $\rho \geq \sqrt{2/\pi} e^{-1/2}$ then $F_\rho$ is monotone increasing, and its supremum is $\rho$, whereas if $0 < \rho < \sqrt{2/\pi} e^{-1/2}$ then $F_\rho$ has a local maximum at the smaller root (i.e. the root in $(0,1)$) of $\sqrt{2/\pi} \lambda e^{-\lambda^2/2} = \rho$; its supremum is the greater of $\rho$ and its value at this maximum. It follows that if $0 < \lambda < 1$ and $\rho = \rho(\lambda) = \sqrt{2/\pi} \lambda e^{-\lambda^2/2}$ then $M_\rho \leq \max(\rho, F_\rho)$. Hence

$$K_C(R) \geq \sup_{0 < \lambda < 1} \frac{1 - \rho(\lambda)}{\max(\rho(\lambda), F_\rho(\lambda))}.$$

Numerically, we find the value 1.67696 for the expression on the right.

**Complex case.** Now we use the inequality $|f(x) + g(x)|^2 \leq R|f(x)g(x)|$ and write $\psi(x) = \sqrt{R} |(1 + f(x)g(x))/2|$, obtaining

$$M_\rho \leq \sup_{\psi \in \mathcal{L}} n \left( \int |\psi(x)| |x_1| |dr| \right)^2 + \rho \left( 1 - 2 \int |\psi(x)|^2 |dr| \right)^2.$$

The maximum will be attained when $\psi$ is either identically zero or of the form $\psi(x) = \psi_{\lambda}(|x_1|) = \min(1, |x_1|/\lambda)$ for some $\lambda \geq 0$. If we also pass to the
limit \( n \to \infty \), when the distribution of \( n^{1/2}x_1 \) approaches complex standard normal, we obtain

\[
M_x \leq \sup_{0 < c < \infty} \left\{ \int_0^\infty 2r^2 \psi_1(r) e^{-rt} dr \right\}^2 + \rho \left( 1 - 4 \int_t^\infty \psi_1(r)^2 e^{-rt} dr \right)
\]

Evaluating the integrals we obtain \( M_x \leq \sup_{0 < c < \infty} G_\rho(\lambda) \) where

\[
G_\rho(\lambda) = \left\{ \lambda^{-1}(1 - e^{-\lambda^2}) + \int_\lambda^\infty e^{-r^2} dr \right\}^2 + \rho \{ 1 - 2\lambda^2(1 - e^{-\lambda^2}) \}
\]

We find that if \( \rho \geq 1/2 \) then \( G_\rho \) is monotone and has supremum \( \rho \), whereas if \( 0 < \rho < 1/2 \) it attains its supremum at the unique value of \( \lambda \) satisfying

\[
\rho = \theta(\lambda) = \frac{1}{2} \{ 1 - e^{-\lambda^2} + \lambda \int_\lambda^\infty e^{-r^2} dr \}. \text{ Hence we obtain}
\]

\[
K_\rho(C) \geq \sup_{0 < c < \infty} \frac{1 - \theta(\lambda)}{G_\rho(\lambda)}
\]

Numerically, we find that the expression on the right is 1.33807 to 5 decimal places.