It's not clear that the following is essentially different from Gunderson-Rödl-Sidorenko. Anyway:

Proposition 1 *Let $A \subseteq \{0,1\}^n$ have density δ. Let $Y_1, \ldots, Y_d$ be a partition of $[n]$ with $|Y_i| \geq r$ for each i. If*

$$\delta^{2^d} - \frac{d}{\sqrt{\pi r}} > 0, \tag{1}$$

then A contains a nondegenerate combinatorial subspace of dimension d, with its ith wildcard set a subset of Y_i.

Proof: Let C_i denote a random chain from $0^{|Y_i|}$ up to $1^{|Y_i|}$, thought of as residing in the coordinates Y_i, with the d chains chosen independently. Also, let s_i, t_i denote independent Binomial$(|Y_i|, 1/2)$ random variables, $i \in [d]$. Note that $C_i(s_i)$ and $C_i(t_i)$ are (dependent) uniform random strings in $\{0,1\}^{Y_i}$. We write, say,

$$(C_1(s_1), C_2(t_2), C_3(t_3), \ldots, C_d(s_d)) \tag{2}$$

for the string in $\{0,1\}^n$ formed by putting $C_1(s_1)$ into the Y_1 coordinates, $C_2(t_2)$ into the Y_2 coordinates, etc. Note that each string of this form is also uniformly random, since the chains are independent.

If all 2^d strings of the form in (2) are simultaneously in A then we have a d-dimensional subspace inside A with wildcard sets that are *subsets* of $Y_1, \ldots, Y_d$. All d dimensions are nondegenerate iff $s_i \neq t_i$ for all i. Since s_i and t_i are independent Binomial$(|Y_i|, 1/2)$'s with $|Y_i| \geq r$, we have

$$\Pr[s_i = t_i] \leq \frac{1}{\sqrt{\pi r}}.$$

Thus to complete the proof, it suffices to show that with probability at least δ^{2^d}, all 2^d strings of the form in (2) are in A.

This is easy: writing f for the indicator of A, the probability is

$$\mathop{\mathbf{E}}_{C_1,\ldots,C_d}\left[\mathop{\mathbf{E}}_{s_1,\ldots,t_d}[f(C_1(s_1),\ldots,C_d(s_d)) \cdots f(C_1(t_1),\ldots,C_d(t_d))]\right].$$

Since $s_1, \ldots, t_d$ are independent, the inside expectation-of-a-product can be changed to a product of expectations. But for fixed $C_1, \ldots, C_d$, each string of the form in (2) has the same distribution. Hence the above equals

$$\mathop{\mathbf{E}}_{C_1,\ldots,C_d}\left[\mathop{\mathbf{E}}_{s_1,\ldots,s_d}[f(C_1(s_1),\ldots,C_d(s_d))]^{2^d}\right].$$

By Jensen (or repeated Cauchy-Schwarz), this is at least

$$\left(\mathop{\mathbf{E}}_{C_1,\ldots,C_d} \mathop{\mathbf{E}}_{s_1,\ldots,s_d} [f(C_1(s_1),\ldots,C_d(s_d))] \right)^{2^d}.$$

But this is just δ^{2^d}, since $(C_1(s_1),\ldots,C_d(s_d))$ is uniformly distributed. □

As an aside:

Corollary 2 *If $A \subseteq [n]$ has density $\Omega(1)$, then A contains a nondegenerate combinatorial subspace of dimension at least $\log_2 \log n - O(1)$.*

If we are willing to sacrifice significantly more probability, we can find a d-dimensional subspace randomly.

Corollary 3 *In the setting of Proposition 1, assume $\delta < 2/3$ and*

$$r \geq \exp(4\ln(1/\delta)2^d). \tag{3}$$

Suppose we choose a random nondegenerate d-dimensional subspace of $[n]$ with wildcard sets $Z_i \subseteq Y_i$. By this we mean choosing, independently for each i, a random combinatorial line within $\{0,1\}^{Y_i}$, uniformly from the $3^r - 1$ possibilities. Then this subspace is entirely contained within A with probability at least 3^{-dr}.

This follows immediately from Proposition 1: having r as in (3) achieves (1), hence the desired nondengenerate combinatorial subspace exists and we pick it with probability $1/(3^r - 1)^d$.

We can further conclude:

Corollary 4 *Let $A \subseteq \{0,1\}^n$ have density $\delta < 2/3$ and let $Y_1,\ldots,Y_d$ be disjoint subsets of $[n]$ with each $|Y_i| \geq r$,*

$$r \geq \exp(4\ln(1/\delta)2^d).$$

Choose a nondegenerate combinatorial subspace at random by picking uniformly nondegenerate combinatorial lines in each of $Y_1,\ldots,Y_d$, and filling in the remaining coordinates outside of the Y_i's uniformly at random. Then with probability at least $\exp(-r^{O(1)})$, this combinatorial subspace is entirely contained within A.

This follows because for a random choice of the coordinates outside the Y_i's, there is a $\delta/2$ chance that A has density at least $\delta/2$ over the Y coordinates. We then apply the previous corollary, noting that $\exp(-r^{O(1)}) \ll (\delta/2)3^{-dr}$, even with δ replaced by $\delta/2$ in the lower bound demanded of r.