Here is a simplified version of the GRS theorem, with slightly better parameters. It gives no improvement for our DHJ results, but might be worth including to illustrate some ideas.

Theorem 0.1. *Let $A \subseteq \{0,1\}^n$ have uniform density $\mu(A) = \delta \le 1/2$ and assume $n \ge 30 \cdot (1/\delta)^{2^d}$. Then A contains a d-dimensional subspace.*

Proof. (Sketch.) Partition $[n] = I_1 \cup I_2 \cup \cdots \cup I_{d-1} \cup E$, where $|I_j| = \lfloor n/4^{d-j} \rfloor$. It follows that $|E| \ge (2/3)n$.

Let C be a random chain over $\{0,1\}^{I_1}$, from 0^{I_1} up to 1^{I_1}. Let s and t be i.i.d. Binomial$(|I_1|, 1/2)$ random variables. We write $C(s)$ for the sth string in chain C. Thus $C(s)$ and $C(t)$ are each uniformly random strings in $\{0,1\}^{I_1}$; they are not independent though, since they always form a (possibly degenerate) line. We write $A_{C(s)}$ for $\{y \in \{0,1\}^{\overline{I_1}} : (C(s), y) \in A\}$. Let z be a uniformly random string in $\{0,1\}^{\overline{I_1}}$. We have

$$\mathop{\mathbf{E}}_{C,s,t}[\mu(A_{C(s)} \cap A_{C(t)})] = \mathop{\mathbf{Pr}}_{C,s,t,z}[(C(s),z),(C(t),z) \in A] = \mathop{\mathbf{E}}_{C,z}\left[\mathop{\mathbf{Pr}}_{s,t}[(C(s),z),(C(t),z) \in A]\right]$$
$$= \mathop{\mathbf{E}}_{C,z}\left[\mathop{\mathbf{Pr}}_{s}[(C(s),z) \in A]^2\right] \ge \mathop{\mathbf{E}}_{C,z}\left[\mathop{\mathbf{Pr}}_{s}[(C(s),z) \in A]\right]^2 = \mathop{\mathbf{Pr}}_{C,s,z}[(C(s),z) \in A]^2 = \mathop{\mathbf{Pr}}_{x \sim \mu}[x \in A] = \delta^2,$$

where on the second line we used the fact that s and t are i.i.d., and then used Cauchy-Schwarz. We have

$$\mathbf{Pr}[s = t] = \mathbf{Pr}[\mathrm{Bin}(2|I_1|, 1/2) = |I_1|] \le \frac{1}{\sqrt{n/4^{d-1}}} = \frac{2^{d-1}}{\sqrt{n}},$$

and hence

$$\mathop{\mathbf{E}}_{C,s,t}[\mu(A_{C(s)} \cap A_{C(t)}) \mid s \ne t] \ge \delta^2 - \frac{2^{d-1}}{\sqrt{n}}.$$

We can therefore fix distinct strings $x_1^{(0)} \prec x_1^{(1)} \in \{0,1\}^{I_1}$ and pass to a subset $A' \subseteq \{0,1\}^{\overline{I_1}}$ such that $(x_1^{(i)}, z) \in A$ for all $i = 0, 1$, $z \in A'$, and such that $\mu(A') \ge \delta^2 - 2^{d-1}/\sqrt{n}$.

We now repeat this argument for the index set I_2. This will give us another pair of distinct strings $x_2^{(0)} \prec x_2^{(1)} \in \{0,1\}^{I_2}$, and let us pass to a subset $A' \subseteq \{0,1\}^{\overline{I_1 \cup I_2}}$ with

$$\mu(A') \ge \left(\delta^2 - \frac{2^{d-1}}{\sqrt{n}}\right)^2 - \frac{2^{d-2}}{\sqrt{n}} \ge \delta^4 - 2\delta^2 \cdot \frac{2^{d-1}}{\sqrt{n}} - \frac{2^{d-2}}{\sqrt{n}} \ge \delta^4 - \frac{2^{d-1}}{\sqrt{n}}.$$

Continuing the argument we get $x_3^{(0)} \prec x_3^{(1)}$ and

$$\mu(A') \ge \left(\delta^4 - \frac{2^{d-1}}{\sqrt{n}}\right)^2 - \frac{2^{d-3}}{\sqrt{n}} \ge \delta^8 - 2\delta^4 \cdot \frac{2^{d-1}}{\sqrt{n}} - \frac{2^{d-3}}{\sqrt{n}} \ge \delta^8 - \frac{2^{d-2}}{\sqrt{n}}.$$

Etc. Having repeated the argument $d-1$ times, we have a $(d-1)$-dimensional subspace $X \subseteq \{0,1\}^{I_1 \cup \cdots \cup I_{d-1}}$ and a subset $A' \subseteq \{0,1\}^E$, where $(x,y) \in A$ for all $x \in X$, $y \in A'$, and $\mu(A') \ge \delta^{2^{d-1}} - 4/\sqrt{n}$. Now if this last quantity is at least $1/\sqrt{|E|}$ then A' contains a line by Sperner's theorem, and we consequently get a d-dimensional subspace in A. But

$$\frac{1}{\sqrt{|E|}} \le \frac{1}{\sqrt{(2/3)n}} \le \frac{1.3}{\sqrt{n}},$$

and even this quantity is at most $\delta^{2^{d-1}} - 4/\sqrt{n}$, because $n \ge (5.3)^2/\delta^{2^d}$ by our initial assumption. □