In this homework, you may wish to consult Lecture 4. Also, you may take for granted the following result, which you basically proved in Homework 9.1:

**Theorem.** Let \( f : \{0, 1 \}^n \to \{0, 1 \} \), let \( \varepsilon \in \left[ \frac{1}{n^3}, 1 \right] \), and let \( \rho \) be an \( \varepsilon \)-random restriction. (Recall this means each coordinate is independently set to ‘*’ (unfixed) with probability \( \varepsilon \), and is otherwise set to 0 or 1 with probability \( \frac{1-\varepsilon}{2} \) each.) Then

\[
E[L(f|\rho)] \leq 2^{1.5}L(f) + 1,
\]

where, recall, \( L(g) \) is the minimum size of a Boolean formula computing \( g \).

(In fact, Johan Håstad and Avishay Tal have shown that \( E[L(f|\rho)] \leq O(\varepsilon^2)L(f) + O(1) \).)

1. **(More on shrinking formulas.)** Let \( b > 1 \), \( m = 2^b \), \( n = bm \). Given some Boolean function \( \psi : \{0, 1 \}^b \to \{0, 1 \} \), define the function \( f_\psi : \{0, 1 \}^n \to \{0, 1 \} \) as follows: Think of \( x \in \{0, 1 \}^n \) as being divided into \( b \) “blocks” of \( m \) bits each. Then \( f_\psi(x) = \psi(z_1, \ldots, z_b) \), where \( z_i \) is the parity (XOR) of the \( i \)th block of bits in \( x \).

   (a) Let \( \varepsilon = \frac{b \ln(3b)}{n} \) and let \( \rho \) be an \( \varepsilon \)-random restriction on \( n = bm \) variables. Show that with probability at least 2/3, the restriction \( \rho \) gives at least one \( * \) to each of the \( b \) blocks.

   (b) Show that there exists a restriction \( \sigma \) of the \( n \) coordinates such that both of the following hold: (i) \( L(f_\psi|\sigma) \leq 6\left(\frac{b \ln(3b)}{n}\right)^{1.5}L(f_\psi) + 3 \); (ii) \( \sigma \) gives at least one \( * \) to each of the \( b \) blocks.

   (c) Show that \( L(f_\psi) \geq \tilde{\Omega}(n^{1.5})(L(\psi) − O(1)) \). Deduce that there exists \( \psi \) such that \( L(f_\psi) \geq \tilde{\Omega}(n^{2.5}) \).

2. **(Andreev’s function.)**

   (a) Does the function \( L_\psi \) produced in the previous problem count as “explicit”?\(^1\) Anyway, let us define an explicit function \( \alpha : \{0, 1 \}^{n+m} \to \{0, 1 \} \), as follows: \( \alpha(x, y) = f_y(x) \), where \( x \in \{0, 1 \}^n \), \( y \in \{0, 1 \}^m \) is interpreted as the truth-table of a function \( \{0, 1 \}^b \to \{0, 1 \} \), and \( f_y \) refers to the “\( f_\psi \)” notation from the previous question. Show that \( L(\alpha) \geq \tilde{\Omega}(n^{2.5}) \).

**Remark.** Using the Håstad–Tal result, one can deduce that in fact \( L(\alpha) \geq n^3/\tilde{O}(\log^3(n)) \).

   (b) Show that \( L(\alpha) \leq O(n^3/\log^2 n) \). (Bonus: show that \( L(\alpha) \leq O(n^3/\log^3 n) \).)

3. **(Detecting triangles.)** Prove that any monotone circuit that detects whether a \( v \)-vertex graph (given by its \( v \times v \) adjacency matrix) contains a triangle must have size at least \( v^3/\text{polylog}(v) \). (Hint: complete bipartite graphs contain no triangles.)

\(^1\)This is a rhetorical question; you are not required to provide an answer.