Graduate Computational Complexity Theory **CMU 15-855*, Fall 2017**

HOMEWORK 10

Due: 10:00am, Tuesday November 21

In this homework, you may wish to consult Lecture 4. Also, you may take for granted the following result, which you basically proved in Homework 9.1:

Theorem. *Let $f : \{0,1\}^n \to \{0,1\}$, let $\varepsilon \in [\frac{1}{n}, 1]$, and let ρ be an ε-random restriction. (Recall this means each coordinate is independently set to '$\star$' (unfixed) with probability ε, and is otherwise set to 0 or 1 with probability $\frac{1-\varepsilon}{2}$ each.) Then*

$$\mathbf{E}[L(f|_\rho)] \le 2\varepsilon^{1.5} L(f) + 1,$$

where, recall, $L(g)$ is the minimum size of a Boolean formula computing g.

(In fact, Johan Håstad and Avishay Tal have shown that $\mathbf{E}[L(f|_\rho)] \le O(\varepsilon^2)L(f) + O(1)$.)

1. **(More on shrinking formulas.)** Let $b > 1$, $m = 2^b$, $n = bm$. Given some Boolean function $\psi : \{0,1\}^b \to \{0,1\}$, define the function $f_\psi : \{0,1\}^n \to \{0,1\}$ as follows: Think of $x \in \{0,1\}^n$ as being divided into b "blocks" of m bits each. Then $f_\psi(x) = \psi(z_1, \dots, z_b)$, where z_i is the parity (XOR) of the ith block of bits in x.

   (a) Let $\varepsilon = \frac{b \ln(3b)}{n}$ and let ρ be an ε-random restriction on $n = bm$ variables. Show that with probability at least 2/3, the restriction ρ gives at least one $\star$ to each of the b blocks.

   (b) Show that there exists a restriction σ of the n coordinates such that both of the following hold: (i) $L(f_\psi|_\sigma) \le 6(\frac{b\ln(3b)}{n})^{1.5} L(f_\psi) + 3$; (ii) σ gives at least one $\star$ to each of the b blocks.

   (c) Show that $L(f_\psi) \ge \widetilde{\Omega}(n^{1.5})(L(\psi) - O(1))$. Deduce that there exists ψ such that $L(f_\psi) \ge \widetilde{\Omega}(n^{2.5})$.

2. **(Andreev's function.)**

   (a) Does the function L_ψ produced in the previous problem count as "explicit"?[1] Anyway, let us define an explicit function $\alpha : \{0,1\}^{n+m} \to \{0,1\}$, as follows: $\alpha(x, y) = f_y(x)$, where $x \in \{0,1\}^n$, $y \in \{0,1\}^m$ is interpreted as the truth-table of a function $\{0,1\}^b \to \{0,1\}$, and f_y refers to the "f_ψ" notation from the previous question. Show that $L(\alpha) \ge \widetilde{\Omega}(n^{2.5})$.

   Remark. *Using the Håstad–Tal result, one can deduce that in fact $L(\alpha) \ge n^3/\widetilde{O}(\log^3(n))$.*

   (b) Show that $L(\alpha) \le O(n^3/\log^2 n)$. (Bonus: show that $L(\alpha) \le O(n^3/\log^3 n)$.)

3. **(Detecting triangles.)** Prove that any monotone circuit that detects whether a v-vertex graph (given by its $v \times v$ adjacency matrix) contains a triangle must have size at least $v^3/\text{polylog}(v)$. (Hint: complete bipartite graphs contain no triangles.)

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[1]This is a rhetorical question; you are not required to provide an answer.