Homework 9
Due: 10:00am, Tuesday November 14

Recall that a Boolean formula F is a binary tree, where the internal nodes are labeled with $\vee$ or $\wedge$, and the leaves are labeled by either literals x_i or $\overline{x}_i$, or by constants 0 or 1. It computes a Boolean function f in the natural way. The *size* of a Boolean formula is the number of literal-leaves. (Constant-leaves are "free".) The least possible size of a formula computing f is denoted $L(f)$.

You may take it for granted that there is a "simplification" operation on formulas that: (i) preserves the function being computed; (ii) gets rid of all constant-leaves (except when the function is itself a constant function, in which case the formula becomes a single constant-leaf); (iii) does not increase the size of the formula. This simplification operation just does the obvious thing: if a 1 enters into an $\vee$ gate, the gate is replaced by 1; if a 1 enters into an $\wedge$ gate, the gate is replaced by its other child; similarly for 0's.

1. **(Random restrictions shrink formulas.)**

   (a) Argue that if F is a Boolean formula, there is an equivalent Boolean formula F' of no larger size with the following property: in F', whenever some internal node has one child a literal $x_i/\overline{x}_i$ and the other child a subformula G, the literals $x_i/\overline{x}_i$ do not appear in G.

   (b) Suppose f is an n-variable Boolean function with $L(f) > 1$. Let ρ be a random restriction formed by fixing exactly one (randomly chosen) variable (to a uniformly random 0/1 value). Show that $\mathbf{E}[L(f|_\rho)] \le (1 - \frac{1.5}{n}) \cdot L(f)$. (Hint: getting $(1 - \frac{1}{n}) \cdot L(f)$ should be easy. If a variable gets fixed, think about what might happen to its sibling-subformulas. Technically, you will need part (a) here.)

   (c) Taking for granted that $(1 - \frac{1.5}{n}) \le (1 - \frac{1}{n})^{1.5}$, show the following: If f is an n-variable Boolean function, and ρ is a random restriction formed by fixing exactly $n-k$ (randomly chosen) variables, then
   $$\mathbf{E}[L(f|_\rho)] \le \max\left\{\left(\frac{k}{n}\right)^{1.5} \cdot L(f), 1\right\}.$$

   (d) Use this (i.e., no fair citing problem 3(c)), with $k = 2$, to prove that $L(\text{Parity}_n) \ge n^{1.5}$.

2. **(Alice and Bob and Parity I.)** A *rectangle* is a set $A \times B$, where $A, B \subseteq \{0,1\}^n$ are disjoint. For $i \in [n]$, we call it *i-colorable* if $x_i \ne y_i$ for all pairs of strings $x \in A$, $y \in B$; we call it *colorable* if it is i-colorable for some i. If a rectangle R can *partitioned* into s (sub)rectangles, each of which is colorable, we say that R is s-tileable. We write $\chi(R)$ for the least s such that R is s-tileable. If f is a Boolean function, we write $\chi(f)$ for $\chi(f^{-1}(0) \times f^{-1}(1))$.

   (a) Prove that $\chi(\text{Parity}_2) = 4$ and $\chi(\text{And}_3) = 3$; draw figures to illustrate the upper bounds.

   (b) Prove that $\chi(f) \le L(f)$. (Hint: induction.)

3. **(Alice and Bob and Parity II.)** Continuing the previous problem...

   (a) Let $R = f^{-1}(0) \times f^{-1}(1)$. Say we "mark" each entry $(x, y) \in R$ where the Hamming distance between x and y is 1. For a subrectangle $A \times B$ of R, write $M(A \times B)$ for the number of marked entries in it. Show that if $A \times B$ is colorable, then $M(A \times B) \le \min\{|A|, |B|\} \le \sqrt{|A| \cdot |B|}$.

(b) Show that $M(R) \leq \sqrt{\chi(f)}\sqrt{|f^{-1}(0)| \cdot |f^{-1}(1)|}$.

(c) Show that $L(\text{Parity}_n) \geq n^2$.

(d) Show that $L(\text{Parity}_n) = n^2$ when n is a power of 2.