Homework 4
Due: 10:00am, Tuesday October 3

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1. **(Could EXP have poly-size circuits?)** We know that $\mathsf{EXP} \not\subseteq \mathsf{P}$ by the Time Hierarchy Theorem. And we sometimes think of P as being roughly comparable to P/poly. Now the latter contains undecidable languages, so they're definitely not the same. Still, it doesn't seem so likely that getting to use a different poly-time algorithm for each input length would be especially helpful for solving an EXP-complete language (like SUCCINCT-CIRCUIT-EVAL, or GENERALIZED-CHESS). But can we get more convincing evidence that $\mathsf{EXP} \subseteq \mathsf{P}/\text{poly}$ is indeed unlikely?

   (a) Let M be a deterministic 1-tape Turing Machine running in time at most 2^{cn^c}. Appropriately formalize — and then also prove — a statement encapsulating the idea that the entries in $M(x)$'s "computation tableau" are computable in EXP. (You might want to look at Sipser's Theorem 7.37 (Cook–Levin) for some terminology and inspiration.)

   (b) Prove that $\mathsf{EXP} \subseteq \mathsf{P}/\text{poly} \implies \mathsf{EXP} = \mathsf{PSPACE}$. (Hence $\mathsf{EXP} \subseteq \mathsf{P}/\text{poly}$ is probably not true, because we consider $\mathsf{EXP} = \mathsf{PSPACE}$ unlikely.)

2. **(Circuit characterization of PH.)** Do Exercise 6.17 in Arora–Barak (called Exercise 6.13 in the "draft version").

3. **(HPV in an alternate universe.)** In this problem you are asked to show the following: There exists a language A such that for all (time- and space-)constructible[1] $f(n)$,

$$\mathsf{TIME}^A(f(n)) = \mathsf{SPACE}^A(f(n)).$$

   Here $\mathsf{TIME}^A(f(n))$ means all languages decidable by a multitape, time-$O(f(n))$ oracle Turing Machine with oracle access to A, and $\mathsf{SPACE}^A(f(n))$ is similarly defined. It is very easy to see that the theorem $\mathsf{TIME}(f(n)) \subseteq \mathsf{SPACE}(f(n))$ "relativizes" (namely, $\mathsf{TIME}^B(f(n)) \subseteq \mathsf{SPACE}^B(f(n))$ for every language B and bound $f(n)$), so the goal is to show that there exists A with $\mathsf{SPACE}^A(f(n)) \subseteq \mathsf{TIME}^A(f(n))$.

   We now describe — roughly — the A you will want to use, although we leave it to you to define A completely precisely. Basically, A should be the language of all tuples $(M, D, x, 1^s)$ such that M is a multitape oracle TM, D is a multitape TM running in space 2^n (how can you ensure this?), x is a string, and (this being the main point) $M^{L(D)}(x)$ accepts while using at most s tape cells. After formalizing A, the first step will be to show that A can be computed in space 2^n. (You might want to add some "padding" into the definition of A to help you show a bound of literally 2^n, not $O(2^n)$.) Then complete the proof, using the fact that there is some D^* deciding A...

[1] In fact, the result is known to hold with no constructibility assumptions at all. For your proof I doubt you'll need space-constructibility. However time-constructibility definitely simplifies the proof. And honestly, I don't even know why I'm typing this footnote, because no one cares about non-constructible functions.