1. (Could EXP have poly-size circuits?) We know that $\text{EXP} \not\subseteq \text{P}$ by the Time Hierarchy Theorem. And we sometimes think of $\text{P}$ as being roughly comparable to $\text{P}/\text{poly}$. Now the latter contains undecidable languages, so they’re definitely not the same. Still, it doesn’t seem so likely that getting to use a different poly-time algorithm for each input length would be especially helpful for solving an $\text{EXP}$-complete language (like $\text{SUCCINCT-CIRCUIT-EVAL}$, or $\text{GENERALIZED-CHESS}$). But can we get more convincing evidence that $\text{EXP} \subseteq \text{P}/\text{poly}$ is indeed unlikely?

(a) Let $M$ be a deterministic 1-tape Turing Machine running in time at most $2^{cn^c}$. Appropriately formalize — and then also prove — a statement encapsulating the idea that the entries in $M(x)$’s “computation tableau” are computable in $\text{EXP}$. (You might want to look at Sipser’s Theorem 7.37 (Cook–Levin) for some terminology and inspiration.)

(b) Prove that $\text{EXP} \subseteq \text{P}/\text{poly} \implies \text{EXP} = \text{PSPACE}$. (Hence $\text{EXP} \subseteq \text{P}/\text{poly}$ is probably not true, because we consider $\text{EXP} = \text{PSPACE}$ unlikely.)


3. (HPV in an alternate universe.) In this problem you are asked to show the following: There exists a language $A$ such that for all (time- and space-)constructible\footnote{1In fact, the result is known to hold with no constructibility assumptions at all. For your proof I doubt you’ll need space-constructibility. However time-constructibility definitely simplifies the proof. And honestly, I don’t even know why I’m typing this footnote, because no one cares about non-constructible functions.} $f(n)$,

$$\text{TIME}^A(f(n)) = \text{SPACE}^A(f(n)).$$

Here $\text{TIME}^A(f(n))$ means all languages decidable by a multitape, time-$O(f(n))$ oracle Turing Machine with oracle access to $A$, and $\text{SPACE}^A(f(n))$ is similarly defined. It is very easy to see that the theorem $\text{TIME}(f(n)) \subseteq \text{SPACE}(f(n))$ “relativizes” (namely, $\text{TIME}^B(f(n)) \subseteq \text{SPACE}^B(f(n))$ for every language $B$ and bound $f(n)$), so the goal is to show that there exists $A$ with $\text{SPACE}^A(f(n)) \subseteq \text{TIME}^A(f(n))$.

We now describe — roughly — the $A$ you will want to use, although we leave it to you to define $A$ completely precisely. Basically, $A$ should be the language of all tuples $(M, D, x, 1^s)$ such that $M$ is a multitape oracle TM, $D$ is a multitape TM running in space $2^n$ (how can you ensure this?), $x$ is a string, and (this being the main point) $M^{L(D)}(x)$ accepts while using at most $s$ tape cells. After formalizing $A$, the first step will be to show that $A$ can be computed in space $2^n$. (You might want to add some “padding” into the definition of $A$ to help you show a bound of literally $2^n$, not $O(2^n)$.) Then complete the proof, using the fact that there is some $D^*$ deciding $A$. .