Graduate Computational Complexity Theory **CMU 15-855*, Fall 2017**

Homework 1

Due: 10:00am, Tuesday September 12

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1. **(Valiant's Depth-Reduction Lemma.)** This problem is concerned with (simple) directed acyclic graphs. The "depth" of such a graph $G = (V, E)$ is defined to be the length of the longest path in the graph. A "labeling" of G is a mapping $\ell : V \to \mathbb{N}$. A labeling ℓ is "legal" if $\ell(u) < \ell(v)$ for all directed edges $(u, v) \in E$.

   (a) Show that if G has a legal labeling using at most d distinct labels, then its depth is less than d. Conversely, show that if G has depth less than d, then it has a legal labeling using at most d distinct labels. (Hint for the latter: consider the "canonical labeling", in which $\ell(v)$ equals the length of the longest path ending at v.)

   (b) Suppose we take the canonical labeling of a graph G and consider the labels to be written in binary. For $j = 0, 1, 2, \dots$, let E_j be all edges (u, v) such that the most significant bit where $\ell(u)$ and $\ell(v)$ differ is the jth. Show that if edges E_j are deleted from G, we can get a legal labeling of the new graph by deleting the jth bit from all labels.

   (c) Deduce the following "depth reduction lemma": Let G be a directed acyclic graph with m edges and depth less than d, where $d = 2^k$. Then for any $1 \le r \le k$, one can reduce the depth to less than $d/2^r$ by the deletion of at most $(r/k)m$ edges.

2. **(Block-respecting TMs.)** Given a "block-size" function $B : \mathbb{N} \to \mathbb{N}^+$, we say a multitape TM is "$B(n)$-block-respecting" if, on length-n inputs, all its tapes are divided into contiguous blocks of $B(n)$ cells, and the tape heads only cross block boundaries at times that are integer multiples of $B(n)$. (In other words, in each segment of $B(n)$ time steps, tape heads always stay within a single block of cells.)

   Let M be a k-tape Turing Machine with running time $T(n)$. Let $1 \le B(n) \le T(n)/2$ be a block-size function. Show there is another Turing Machine M' with $O(k)$ tapes[1] that is $B(n)$-block-respecting, decides the same language as M, and has running time $O(T(n))$.[2]

   The TAs will pay extra attention to the *quality of your exposition* in this problem.

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3. **(Improving the Time Hierarchy Theorem via padding.)**

   (a) Let $t_1 : \mathbb{N} \to \mathbb{N}$ be a nondecreasing time-constructible function with $t_1(n) \ge n$, and let t_2 and f be two more such functions.[3] Show that $\mathsf{TIME}(t_1(n)) = \mathsf{TIME}(t_2(n))$ implies $\mathsf{TIME}(t_1(f(n))) = \mathsf{TIME}(t_2(f(n)))$. (Hint: padding.)

   (b) Show that $\mathsf{TIME}(n^3 \log^{3/4} n) \ne \mathsf{TIME}(n^3)$. You may use the Time Hierarchy Theorem (Theorem 3.1 in Arora–Barak), and you may take it for granted that any normal-looking functions are time-constructible. (Hint: you may need to use part (a) *several* times.)

   It is a fact (you don't need to prove it) that a suitable elaboration of this problem shows $\mathsf{TIME}(n^a \log^{\varepsilon} n) \ne \mathsf{TIME}(n^a)$ for all $a \ge 1$ and $\varepsilon > 0$.

Footnotes

[1] $3k + 1$, or even $k + 1$, should be possible.

[2] Technicalities: First, you may assume $B(n)$ has the following "time- and space-constructibility" properties: There is a 2-tape TM that, on inputs of length n, uses $O(T(n))$ time and exactly $B(n)$ space (on its second tape, only reading the input tape), and writes $B(n)$ in unary on the second tape. Further, your M' may use this routine at the beginning, and only *then* become $B(n)$-block-respecting.

[3] We would like to also talk about functions like $\sqrt{t_1}$ or $t_2 \log t_2$ without worrying about the fact that these could be real-valued. Assume that real values arising in such expressions are always rounded up; or, just choose not to worry about it.