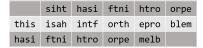
HOMEWORK 1 Due: 10:00am, Tuesday September 12

- 1. (Valiant's Depth-Reduction Lemma.) This problem is concerned with (simple) directed acyclic graphs. The "depth" of such a graph G = (V, E) is defined to be the length of the longest path in the graph. A "labeling" of G is a mapping $\ell: V \to \mathbb{N}$. A labeling ℓ is "legal" if $\ell(u) < \ell(v)$ for all directed edges $(u, v) \in E$.
 - (a) Show that if G has a legal labeling using at most d distinct labels, then its depth is less than d. Conversely, show that if G has depth less than d, then it has a legal labeling using at most d distinct labels. (Hint for the latter: consider the "canonical labeling", in which $\ell(v)$ equals the length of the longest path ending at v.)
 - (b) Suppose we take the canonical labeling of a graph G and consider the labels to be written in binary. For $j = 0, 1, 2, \ldots$, let E_j be all edges (u, v) such that the most significant bit where $\ell(u)$ and $\ell(v)$ differ is the jth. Show that if edges E_j are deleted from G, we can get a legal labeling of the new graph by deleting the jth bit from all labels.
 - (c) Deduce the following "depth reduction lemma": Let G be a directed acyclic graph with m edges and depth less than d, where $d = 2^k$. Then for any $1 \le r \le k$, one can reduce the depth to less than $d/2^r$ by the deletion of at most (r/k)m edges.
- 2. (Block-respecting TMs.) Given a "block-size" function $B : \mathbb{N} \to \mathbb{N}^+$, we say a multitape TM is "B(n)-block-respecting" if, on length-n inputs, all its tapes are divided into contiguous blocks of B(n) cells, and the tape heads only cross block boundaries at times that are integer multiples of B(n). (In other words, in each segment of B(n) time steps, tape heads always stay within a single block of cells.)

Let M be a k-tape Turing Machine with running time T(n). Let $1 \leq B(n) \leq T(n)/2$ be a block-size function. Show there is another Turing Machine M' with O(k) tapes¹ that is B(n)-block-respecting, decides the same language as M, and has running time O(T(n)).

The TAs will pay extra attention to the quality of your exposition in this problem.



- 3. (Improving the Time Hierarchy Theorem via padding.)
 - (a) Let $t_1 : \mathbb{N} \to \mathbb{N}$ be a nondecreasing time-constructible function with $t_1(n) \geq n$, and let t_2 and f be two more such functions.³ Show that $\mathsf{TIME}(t_1(n)) = \mathsf{TIME}(t_2(n))$ implies $\mathsf{TIME}(t_1(f(n))) = \mathsf{TIME}(t_2(f(n)))$. (Hint: padding.)
 - (b) Show that $\mathsf{TIME}(n^3\log^{3/4}n) \neq \mathsf{TIME}(n^3)$. You may use the Time Hierarchy Theorem (Theorem 3.1 in Arora–Barak), and you may take it for granted that any normal-looking functions are time-constructible. (Hint: you may need to use part (a) several times.) It is a fact (you don't need to prove it) that a suitable elaboration of this problem shows $\mathsf{TIME}(n^a\log^\varepsilon n) \neq \mathsf{TIME}(n^a)$ for all $a \geq 1$ and $\varepsilon > 0$.

Footnotes

 $^{^{1}3}k + 1$, or even k + 1, should be possible.

²Technicalities: First, you may assume B(n) has the following "time- and space-constructibility" properties: There is a 2-tape TM that, on inputs of length n, uses O(T(n)) time and exactly B(n) space (on its second tape, only reading the input tape), and writes B(n) in unary on the second tape. Further, your M' may use this routine at the beginning, and only then become B(n)-block-respecting.

³We would like to also talk about functions like $\sqrt{t_1}$ or $t_2 \log t_2$ without worrying about the fact that these could be real-valued. Assume that real values arising in such expressions are always rounded up; or, just choose not to worry about it.