CmpE 587: Intro to Research in TCS **Boğaziçi University, Fall 2014**

Homework 4

Due: November 26, 11:59pm.

Email your PDF as an attachment to cmpe587homework@gmail.com

The attached file should be called `homework4-yourfirstname-yourlastname.pdf`

Homework policy: Please try to do the homework by yourself. If you get stuck, working in a group of two is okay, three at the most. Naturally, acknowledge any sources you worked with at the top of your solutions. LaTeX typesetting with pdf output is mandatory. Questions about the homework, LaTeX, or other course material can be asked on Piazza.

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1. You are back at the manav, with the apple and the banana. Today's market research indicates the following: 1/3 of potential customers are Apple-Eaters; they value an apple at 1₺ and a banana at 0. Another 1/3 of potential customers are Banana-Likers; they value a banana at 2₺ and an apple at 0. Finally, 1/3 of potential customers are Fruit-Lovers; they value an apple at 3₺ and a banana at 3₺. As always, you must post a menu with a fixed price for each of the three "bundles" (apple, banana, apple+banana), and the one random customer that comes to the manav may only make one transaction.

   (a) Find a (deterministic) menu for which your expected revenue is $\frac{7}{3}$₺.

   (b) Now suppose you are allowed "menus with lottery tickets"; that means that for each bundle B, in addition to the price t_B you may also include a probability p_B. If the buyer decides to buy that bundle, they pay t_B₺ and receive the bundle with probability p_B. (In particular, with probability $1 - p_B$ they get nothing, even though they paid t_B₺already.) Find and analyze such a menu for which your expected revenue is $\frac{5}{2}$₺.

   (c) (Open-ended.) Note that $\frac{5}{2} > \frac{7}{3}$. "Why" do you feel you are able to get more revenue by allowing these randomized sales? The more compelling your reasoning/discussion, the more homework points. Bonus points for proving optimality for one or both of the solutions in parts (a), (b).

2. Given as input is an n-vertex undirected graph $G = (V, E)$, as well as an integer $0 \leq t \leq n$. The task is to output the number of different "independent sets" S in G with cardinality $|S|$ exactly equal to t. (Reminder: $S \subseteq V$ is called an "independent set" if there is no edge $e \in E$ with both its endpoints in V.) Fact: For general graphs G, this task is not just "NP-hard", it's even "♯P-hard" (whatever that means; suffice it to say, it's even harder than "NP-hard"). Anyway, solve one of the following:

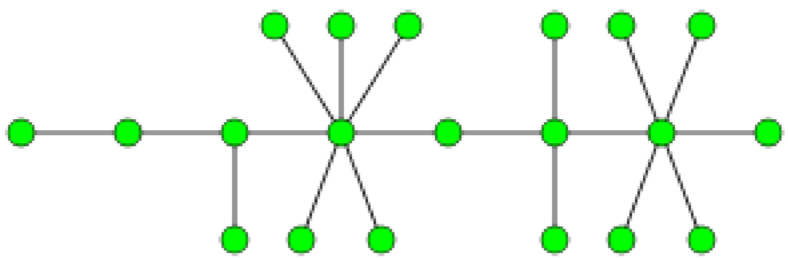
   (a) (For a maximum of 4 out of 5 points.) Show that this task is solvable in poly(n) time if G is promised to be a tree.

   (b) (For a maximum of 5 out of 5 points.) Show that this task is solvable in poly(n) time if G is a series/parallel graph. You may assume that the "series/parallel" construction of G is given to you as part of the input.

   (c) (For a maximum of 6 out of 5 points.) Show that this task is solvable in $n^{O(k)}$ time if G is a graph of treewidth at most k. You may assume that a tree decomposition of width k is given as part of the input.

3. *Pathwidth* is a simpler version of treewidth, which has several algorithmic applications (including to out-of-order execution in compiler design). A *path decomposition* of a graph G is just a tree decomposition in which the tree of bags T is in fact just a path of bags. (Thus, T must a path of bags, where each bag contains a set of vertices of G. For each edge (u, v) of G, both u and v must be together in some bag. And for each vertex v in G, the set of bags of T containing v must be a subpath of T.) As usual, the *width* of a path decomposition is the cardinality of the largest bag, minus one. And the pathwidth of G is the least possible width among all path decompositions of G.

   Assume that G is a connected graph. Prove that G has pathwidth 1 if and only if G is a "caterpillar graph". (A caterpillar graph is a graph which consists of a "central" path, plus optionally some more vertices that have distance one from the central path. An example is below.)



   (A fun fact, which you are not allowed to use in this problem: The pathwidth of G is also one less than the number of cops needed to win the "Cops and Robbers" game on G when the robber is *invisible* to the cops!)

4. Define the function $S : \{\pm 1\}^4 \to \{\pm 1\}$ by $S(x) = -1$ if and only if either $x_1 \geq x_2 \geq x_3 \geq x_4$ or $x_1 \leq x_2 \leq x_3 \leq x_4$. Compute the Fourier expansion of S, and give some indication of your method. (In other words, don't just write the answer.)

5. Recall how the "BLR Subtest" works when applied to $f : \{\pm 1\}^n \to \{\pm 1\}$. First the tester chooses strings $x, y \sim \{\pm 1\}^n$ uniformly at random. Next it forms $z \in \{\pm 1\}^n$ by defining $z_i = x_i y_i$ for all $1 \leq i \leq n$. Next it "queries" f at x, y, and z to determine $f(x), f(y), f(z)$. Finally, it "accepts" if $f(z) = f(x)f(y)$ and "rejects" if $f(z) \neq f(x)f(y)$. Prove that

$$\mathbf{Pr}[\text{the subtest "rejects"}] = \frac{1}{2} - \frac{1}{2} \sum_{S \subseteq \{1,\dots,n\}} \widehat{f}(S)^3.$$

   (Hint: what can you say about the quantity $\mathbf{E}[\frac{1}{2} - \frac{1}{2} f(x)f(y)f(z)]$?)