Let's define a combinatorial *bridge* in $[3]^n$ to be a triple of points $(x, y, z) \in \{-1, 0, 1\}^n$ formed by taking a string with zero or more wildcards and filling in the wildcards with either $(0, -1, 0)$ or $(0, 1, 0)$. If there are zero wildcards we call the bridge degenerate. I think I can show, using the ideas from #800, that if $f : \{-1, 0, 1\}^n \to \{0, 1\}$ has mean $\delta > 0$ and $n$ is sufficiently large as function of $\delta$, then there is a nondegenerate bridge $(x, y, z)$ with $f(x) = f(y) = f(z) = 1$.

We'll actually work with a slightly 0-biased distribution. Let $\epsilon = \epsilon(\delta, n)$ be a small quantity to be named later, and define $\mu$ to be the product distribution on $\{-1, 0, 1\}^n$ where each coordinate is 0 with probability $1/3 + 2\epsilon/3$ and $\pm 1$ with probability $1/3 - \epsilon/3$. Similar to in the document from #800, if $\epsilon \ll \sqrt{\delta/n}$ then we still have $\mathbf{E}_\mu[f] \geq \delta/2$.

Let $\tilde{f}$ denote the "balanced" version of $f$, i.e., $f - \mathbf{E}_\mu[f]$. Again, as in #800, we can run a density-increment argument, making progress whenever $\|T^\mu_{1-\gamma}\tilde{f}\|_2^2$ is even a teeny bit bigger than $\mathbf{E}_\mu[f]^2$. This teeny bit can be any small function of $\delta$ and $\gamma > 0$ can be any function of $\delta$ and $n$ that is noticeably bigger than $1/n$. In other words, we can assume $\tilde{f}$ is extraordinarily noise-sensitive.

Now let's pick a distribution on bridges. Let $(x_1, y_1, z_1) \in \{-1, 0, 1\}^3$ be chosen jointly as follows: with probability $1 - \epsilon$ we set $x_1 = y_1 = z_1 =$ a uniformly random digit in $\{-1, 0, 1\}$; with probability $\epsilon$, we set $(x_1, y_1, z_1)$ to be one of $\{(0, -1, 0), (0, 0, 0), (0, 1, 0)\}$, with equal probability. (We're sneakily allowing $(0, 0, 0)$ here.) Finally, let $(x, y, z)$ be a triple of strings formed by choosing each $(x_i, y_i, z_i)$ as we chose $(x_1, y_1, z_1)$, independently across $i$. Note that $(x, y, z)$ is always a bridge, and is nondegenerate whp assuming $\epsilon$ is not ridiculously small. Further, note that the $x$ and $z$ marginals are $\mu$, whereas the $y$ marginal is uniform.

We would like to show $\mathbf{E}[f(x)f(y)f(z)]$ is $\Omega(\delta^3)$. Passing to the balanced versions $\tilde{f}$ on $x$ and $z$ and the uniform-distribution-balanced version $\tilde{\tilde{f}}$ on $y$, it suffices to show that

$$\delta \gg \delta\mathbf{E}[\tilde{f}(x)\tilde{\tilde{f}}(y)] + \delta\mathbf{E}[\tilde{f}(x)\tilde{f}(z)] + \delta\mathbf{E}[\tilde{\tilde{f}}(y)\tilde{f}(z)] + \mathbf{E}[\tilde{f}(x)\tilde{\tilde{f}}(y)\tilde{f}(z)].$$

I'll show the last term is small, and the other three are easier. The point is that there is "imperfect correlation" between the random variable $(x, y)$ and the random variable $z$ — and similarly, between $x$ and $(y, z)$. Here I use the term in the sense of the Mossel paper in #811. If I calculate correctly, the correlation is $\sqrt{1-\epsilon} \approx 1 - \epsilon/2$. At any rate, we can now appeal to Lemma 6.2 from Mossel's paper to show that

$$\mathbf{E}[\tilde{f}(x) \cdot \tilde{\tilde{f}}(y) \cdot \tilde{f}(z)] = \mathbf{E}[T^\mu_{1-\gamma}\tilde{f}(x) \cdot \tilde{\tilde{f}}(y) \cdot T^\mu_{1-\gamma}\tilde{f}(z)] \pm O(\mu^4),$$

say, assuming $\gamma$ is taken smaller than $\delta^{O(1)}\epsilon = \delta^{O(1)}/\sqrt{n}$. But now we're practically done; by Hölder we have

$$|\mathbf{E}[T^\mu_{1-\gamma}\tilde{f}(x) \cdot \tilde{\tilde{f}}(y) \cdot T^\mu_{1-\gamma}\tilde{f}(z)]| \leq \|T^\mu_{1-\gamma}\tilde{f}(x)\|_3 \cdot \|\tilde{\tilde{f}}(y)\|_3 \cdot \|T^\mu_{1-\gamma}\tilde{f}(z)\|_3.$$

And actually the first and third factors can both be made small: by boundedness we have $\|T^{\mu}_{1-\gamma}\tilde{f}(x)\|_3 \leq (\|T^{\mu}_{1-\gamma}\tilde{f}(x)\|_2^2)^{1/3}$, and we know we can make $\|T^{\mu}_{1-\gamma}\tilde{f}(x)\|_2^2$ arbitrarily small by the density-increment argument.