1 The rationality of weakly symmetric social choice functions

Lemma 1.1 *Let* $f : \{-1,1\}^n \to \mathbb{R}$ *be weakly symmetric (transitive). Then* $\hat{f}(i)$ *is the same for all* $i \in [n]$.

Proof: Let $i \neq i'$. Since f is weakly symmetric, we can pick a permutation π on $[n]$ such that $\pi(i) = i'$ and

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = f(x_1, \ldots, x_n)$$

for all $x \in \{-1,1\}^n$. Now

$$\hat{f}(i) = \mathbf{E}[f(\boldsymbol{x})\boldsymbol{x}_i] = \mathbf{E}[f(\boldsymbol{x}_{\pi(1)}, \ldots, \boldsymbol{x}_{\pi(n)})\boldsymbol{x}_{\pi(i)}],$$

since a uniformly random string hit by a permutation is still uniformly random. But the above equals $\mathbf{E}[f(\boldsymbol{x})\boldsymbol{x}_{i'}] = \hat{f}(i')$. $\square$

Proposition 1.2 (Kalai 2002) *If* $f : \{-1,1\}^n \to \{-1,1\}$ *is a weakly symmetric social choice function,* $\mathrm{Rationality}(f) \leq \frac{7}{9} + \frac{4}{9\pi} + O(1/n) \approx .919$.

Proof: Since f is weakly symmetric, the lemma implies that $\hat{f}(i)$ is the same for all i's. Hence

$$W_1(f) = \sum_{i=1}^{n} \hat{f}(i)^2 = \frac{1}{n}\left(\sum_{i=1}^{n} \hat{f}(i)\right)^2.$$

But we proved that for any f, $\sum_{i=1}^{n} \hat{f}(i)$ is at most that of Majority, namely $\sqrt{2/\pi}\sqrt{n}+O(1/\sqrt{n})$. Hence

$$W_1(f) \leq \frac{2}{\pi} + O(1/n),$$

and the result follows from our proposition bounding $\mathrm{Rationality}(f)$ in terms of $W_1(f)$. $\square$