1 Håst-Odd test: Dictatorship versus Quasirandom

Theorem 1.1 *Let* $h : \{-1,1\}^n \to [-1,1]$ *be* (ϵ^2, δ)*-quasirandom. Then*

$$\mathbf{Pr}[\textit{Håst-Odd}_\delta(h) \textit{ passes}^1] < \tfrac{1}{2} + \tfrac{1}{2}\epsilon.$$

Proof: By contraposition:

$$\begin{aligned}
\frac{1}{2} + \frac{1}{2}\epsilon &\leq \frac{1}{2} + \frac{1}{2}\sum_{|S| \text{ odd}} (1-2\delta)^{|S|}\hat{h}(S)^3 \\
\Rightarrow \qquad \epsilon &\leq \sum_{|S| \text{ odd}} (1-2\delta)^{|S|}\hat{h}(S)^3 \\
&\leq \max_{|S| \text{ odd}} \{(1-2\delta)^{|S|}\hat{h}(S)\} \cdot \sum_{|S| \text{ odd}} \hat{h}(S)^2 \\
&\leq \max_{|S| \text{ odd}} \{(1-2\delta)^{|S|}\hat{h}(S)\},
\end{aligned}$$

since $\sum_{|S| \text{ odd}} \hat{h}(S)^2 \leq \sum_{S \subseteq [n]} \hat{h}(S)^2 = \mathbf{E}[h^2] \leq 1$. We conclude that there is some $S^* \subseteq [n]$ with $|S^*|$ odd such that

$$(1-2\delta)^{|S^*|}\hat{h}(S^*) \geq \epsilon \qquad \Rightarrow \qquad (1-2\delta)^{2|S^*|}\hat{h}(S^*)^2 \geq \epsilon^2; \tag{1}$$

in particular, since $|S^*|$ is odd, $S^* \neq \emptyset$. Choosing any $i \in S^*$, we have

$$\mathrm{Inf}_i^{(1-\delta)}(h) = \sum_{S \ni i} (1-\delta)^{|S|-1}\hat{h}(S)^2 \geq (1-\delta)^{|S^*|-1}\hat{h}(S^*)^2 \geq (1-2\delta)^{2|S^*|}\hat{h}(S^*)^2 \geq \epsilon^2.$$

This means h is not (ϵ^2, δ)-quasirandom. □

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[1] More accurately, if $h = \mathrm{avg}\{f_1, \ldots, f_d\}$ for some boolean-valued functions $f_i : \{-1,1\}^n \to \{-1,1\}$, then the probability the test passes when applied to the collection.