Analysis of Boolean Functions **CMU 18-859S / 21-801A, Fall 2012**

PROBLEM SET 6

Due: Wednesday, Oct. 24, beginning of class

Homework policy: Questions about the homework or other course material can be asked on Piazza.

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1. Let $A \subseteq \{-1,1\}^n$ have cardinality $\alpha 2^n$, $\alpha \le 1/2$. Thinking of $\{-1,1\}^n \subset \mathbb{R}^n$, let $\mu_A \in \mathbb{R}^n$ be the center of mass of A. Show that μ_A is close to the origin in Euclidean distance: $\|\mu_A\|_2 \le O(\sqrt{\log(1/\alpha)})$. (Hint: Level 1 Inequality.)

2. (Chang's Lemma.) Let $A \subseteq \mathbb{F}_2^n$ have $|A|/2^n = \alpha \le 1/2$. Let φ_A denote the probability density function associated to the uniform distribution on A. Let $\epsilon > 0$ and let $\Gamma = \{\gamma \in \mathbb{F}_2^n : |\widehat{\varphi_A}(\gamma)| \ge \epsilon\}$. (We use the notation from Homework 3, #3.)

   (a) Compute $\mathbf{E}[\varphi_A^2]$ and use Parseval to show that $|\Gamma| \le (1/\epsilon^2)(1/\alpha)$. (The goal of this exercise is to show that $\dim(\mathrm{span}(\Gamma))$ is much smaller.)

   (b) Let $\gamma_1, \dots, \gamma_d$ be a maximal set of linearly independent vectors in Γ. Let $M \in \mathbb{F}_2^{n\times n}$ be an invertible matrix whose first d rows are $\gamma_1, \dots, \gamma_d$. Let $A' = MA = \{Mx : x \in A\}$ and let $\varphi_{A'}$ be the probability density function for the uniform distribution on A'. What can you say about $\widehat{\varphi_{A'}}(0)$ and $\widehat{\varphi_{A'}}(e_1), \dots, \widehat{\varphi_{A'}}(e_d)$? (Here $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, with the 1 in the ith position.)

   (c) Show that $d = \dim(\mathrm{span}(\Gamma)) \le O((1/\epsilon)^2 \log(1/\alpha))$.

3. Let $\boldsymbol{X} \not\equiv 0$ be a B-reasonable random variable; i.e., $\mathbf{E}[\boldsymbol{X}^4] \le B\,\mathbf{E}[\boldsymbol{X}^2]^2$; i.e., $\|\boldsymbol{X}\|_4 \le B^{1/4}\|\boldsymbol{X}\|_2$.

   (a) Show that $\mathbf{Pr}[|\boldsymbol{X}| \ge t\|\boldsymbol{X}\|_2] \le B/t^4$ for all $t > 0$. (Hint: apply Markov's Inequality to $\boldsymbol{X}^4$.)

   (b) Let $\boldsymbol{Y}$ be a nonnegative random variable with $0 < \mathbf{E}[\boldsymbol{Y}^2] < \infty$. Show the following "second moment method" inequality: $\mathbf{Pr}[\boldsymbol{Y} > \theta\,\mathbf{E}[\boldsymbol{Y}]] \ge (1-\theta)^2 \frac{\mathbf{E}[\boldsymbol{Y}]^2}{\mathbf{E}[\boldsymbol{Y}^2]}$ for all $0 \le \theta \le 1$. (Hint: start by writing $\mathbf{E}[\boldsymbol{Y}] = \mathbf{E}[\boldsymbol{Y} \cdot \mathbf{1}_{\{\boldsymbol{Y} \le \theta \mathbf{E}[\boldsymbol{Y}]\}}] + \mathbf{E}[\boldsymbol{Y} \cdot \mathbf{1}_{\{\boldsymbol{Y} > \theta \mathbf{E}[\boldsymbol{Y}]\}}]$; use Cauchy–Schwarz on the latter.)

   (c) Show that $\mathbf{Pr}[|\boldsymbol{X}| \ge t\|\boldsymbol{X}\|_2] \ge (1-t^2)^2/B$ for all $0 \le t \le 1$.

4. Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ be independent uniform ± 1 bits and let $a_1, \dots, a_n \in \mathbb{R}$. Show that $a_1\boldsymbol{x}_1 + \cdots + a_n\boldsymbol{x}_n$ is a 3-reasonable random variable.

5. Let $f : \{-1,1\}^n \to \mathbb{R}$. Show that $\|\mathrm{T}_{1/\sqrt{3}} f\|_4 \le \|f\|_2$. (Hint: show $\mathbf{E}[(\mathrm{T}_{1/\sqrt{3}} f(\boldsymbol{x}))^4] \le \mathbf{E}[f(\boldsymbol{x})^2]^2$ by induction on n, similar to our proof of Bonami's Lemma.)