1. Suppose \( f(x) = \text{sgn}(a_0 + a_1x_1 + \cdots + a_nx_n) \) is an LTF with \(|a_1| \geq |a_2| \geq \cdots \geq |a_n|\). Show that \( \text{Inf}_1[f] \geq \text{Inf}_2[f] \geq \cdots \geq \text{Inf}_n[f] \).

2. In class we will discuss the FKN Theorem and the proof of the following: If \( f : \{-1,1\}^n \to \{-1,1\} \) has \( E[f] = 0 \) and \( W^1[f] \geq 1 - \delta \) then \( f \) is \( O(\delta) \)-close to \( \pm \chi_i \) for some \( i \in [n] \). Assuming this, show the following: If \( f : \{-1,1\}^n \to \{-1,1\} \) has \( W^{\leq 1}[f] \geq 1 - \delta \) then \( f \) is \( O(\delta) \)-close to a 1-junta. (Hint: define \( g(x_0, x) = x_0f(x_0, x) \).)

3. (a) Suppose \( \ell : \{-1,1\}^n \to \mathbb{R} \) is defined by \( \ell(x) = a_0 + a_1x_1 + \cdots + a_nx_n \). Define \( \tilde{\ell} : (1,1)^{n+1} \to \mathbb{R} \) by \( \tilde{\ell}(x_0, \ldots, x_n) = a_0x_0 + a_1x_1 + \cdots + a_nx_n \). Show that \( \| \tilde{\ell} \|_1 = \| \ell \|_1 \) and \( \| \tilde{\ell} \|_2^2 = \| \ell \|_2^2 \).

(b) Show that if \( f = \text{sgn}(\ell) \) is any LTF then \( W^{\leq 1}[f] \geq 1/2 \). (Recall that we proved this in class assuming \( a_0 = 0 \) using Homework #2, Problem #6.)

4. Consider the sequence of LTFs \( f_n : \{-1,1\}^n \to \{0,1\} \) defined by \( f_n(x) = 1 \) if and only if \( \sum_{i=1}^n \frac{1}{\sqrt{n}} x_i > t \).

(I.e., \( f_n \) is the indicator of the Hamming ball of radius \( \frac{n}{2} - \frac{t}{2\sqrt{n}} \) centered at \((1,1,\ldots,1)\).) Show that

\[
\lim_{n \to \infty} E[f_n] = \Phi(t), \quad \lim_{n \to \infty} W[f_n] = \phi(t)^2,
\]

where \( \phi \) is the pdf of a standard Gaussian and \( \Phi \) is the complementary cdf (i.e., \( \Phi(u) = \int_u^\infty \phi \)). You may use the Central Limit Theorem without worrying about error bounds.

5. For integer \( 0 \leq k \leq n \), define \( \mathcal{K}_k : \{-1,1\}^n \to \mathbb{R} \) by \( \mathcal{K}_k(x) = \sum_{|S| = k} x_S \). Since \( \mathcal{K}_k \) is symmetric, the value \( \mathcal{K}_k(x) \) depends only on the number of \(-1\)'s in \( x \); or equivalently, on \( \sum_{i=1}^n x_i \). Thus we may define \( \mathcal{K}_k : \{0,1,\ldots,n\} \to \mathbb{R} \) by \( \mathcal{K}_k(z) = \mathcal{K}_k(x) \) for any \( x \) with \( \sum x_i = n - 2z \).

(a) Show that \( \mathcal{K}_k(z) \) can be expressed as a degree-\( k \) polynomial in \( z \). It is called the Kravchuk (or Krawtchouk) polynomial of degree \( k \). (The dependence on \( n \) is usually implicit.)

(b) Show that \( \sum_{k=0}^n \mathcal{K}_k(x) = \begin{cases} 2^n & \text{if } x = (1,\ldots,1), \\ 0 & \text{else.} \end{cases} \)

(c) Show for \( \rho \in [-1,1] \) that \( \sum_{k=0}^n \mathcal{K}_k(x) \rho^k = 2^n \text{Pr}[y = (1,\ldots,1)] \), where \( y = N_\rho(x) \).

(d) Deduce the following generating function identity: \( \mathcal{K}_k(z) = [\rho^k][(1-\rho)^z(1+\rho)^{n-z}] \); i.e., the coefficient on \( \rho^k \) in \( (1-\rho)^z(1+\rho)^{n-z} \).

6. Say that \( \tilde{Z}, \tilde{Z}' \) are "\( \rho \)-correlated \( d \)-dimensional Gaussians" if the pairs \((\tilde{Z}_1, \tilde{Z}'_1), \ldots, (\tilde{Z}_d, \tilde{Z}'_d)\) are independent \( \rho \)-correlated Gaussians. Now given \( f : \mathbb{R}^d \to (-1,1) \) and \( \epsilon \in \mathbb{R} \), define the rotation sensitivity of \( f \) at \( \epsilon \) to be

\[
\text{RS}_f(\epsilon) = \text{Pr}[f(\tilde{Z}) \neq f(\tilde{Z}')],
\]

where \( \tilde{Z} \) and \( \tilde{Z}' \) are cos(\( \epsilon \))-correlated \( d \)-dimensional Gaussians.
(a) Show that $\mathbf{RS}_f(0) = 0$, $\mathbf{RS}_f(\pi/2) = \frac{1}{2}$ if $\Pr[f(\tilde{Z}) = 1] = \frac{1}{2}$, and $\mathbf{RS}_f(\pi) = 1$ if $f$ is an odd function (meaning $f(-x) = -f(x)$).

(b) For $j = 0,1,\ldots,\ell$, let $\tilde{u}_j$ be the unit vector in $\mathbb{R}^2$ making an angle of $j\epsilon$ with the $x$-axis. Let $\tilde{Y}$ be a standard 2-dimensional Gaussian and define $Z_j = \langle \tilde{u}_j, \tilde{Y} \rangle$. Show that $Z_j$ and $Z_{j+k}$ are cos($k\epsilon$)-correlated standard Gaussians.

(c) Describe how to generate a sequence of $d$-dimensional Gaussians $\tilde{Z}^{(0)}, \ldots, \tilde{Z}^{(\ell)}$ such that each pair $\tilde{Z}^{(j)}, \tilde{Z}^{(j+k)}$ is cos($k\epsilon$)-correlated (as defined in the first sentence of this problem).

(d) Show that for any $f : \mathbb{R}^d \to \{-1,1\}$, $\epsilon \in \mathbb{R}$, and $\ell \in \mathbb{N}^+$ we have $\mathbf{RS}_f(\ell \epsilon) \leq \ell \mathbf{RS}_f(\epsilon)$. (Hint: previous part + union bound.)

(e) Let $f : \mathbb{R}^d \to \{-1,1\}$ satisfy $\Pr[f(\tilde{Z}) = 1] = 1/2$ where $\tilde{Z}$ is a $d$-dimensional Gaussian. Let $\epsilon = \frac{\pi}{2\ell}$ for some $\ell \in \mathbb{N}^+$. Show that $\mathbf{RS}_f(\epsilon) \geq \epsilon/\pi$. (This last result is a special case of a 1985 theorem of Borell and is also generalization of a special case of the Gaussian Isoperimetric Inequality...)

7. (Bonus problem due to [JOW’12].) Let $X$ and $Y$ be symmetric random variables (meaning $-X$ has the same distribution as $X$, and similarly for $Y$). Show that $\min(\text{Var}[X], \text{Var}[Y]) \leq C \cdot \text{Var}(|X + Y|)$ for some universal constant $C$. 