1. Compute the Fourier expansions of the following functions.

(a) The selection function $\text{Sel} : \{-1,1\}^3 \rightarrow \{-1,1\}$ which outputs $x_2$ if $x_1 = -1$ and outputs $x_3$ if $x_1 = 1$.

(b) The indicator function $1_{\{a\}} : \{-1,1\}^n \rightarrow \{0,1\}$, where $a \in \{-1,1\}^n$.

(c) The density function corresponding to the product probability distribution on $\{-1,1\}$ in which each coordinate has mean $\rho \in [-1,1]$;

(d) The inner product mod 2 function, $\text{IP}_{2n} : \mathbb{F}_2^n \rightarrow \{-1,1\}$ defined by $\text{IP}_{2n}(x_1, \ldots, x_n, y_1, \ldots, y_n) = (-1)^{x \cdot y}$. (Here $x \cdot y$ denotes the dot-product in the vector space $\mathbb{F}_2^n$.)

(e) The hemi-icosahedron function $\text{HI} : \{-1,1\}^6 \rightarrow \{-1,1\}$, defined as follows: $\text{HI}(x)$ is 1 if the number of 1’s in $x$ is 1, 2, or 6. $\text{HI}(x)$ is $-1$ if the number of $-1$’s in $x$ is 1, 2, or 6. Otherwise, $\text{HI}(x)$ is 1 if and only if one of the ten facets in the following diagram has all three of its vertices 1:

![Figure 1: The hemi-icosahedron](image)

(Please give some indication of how you arrived at the expansion; a bare formula does not suffice.)

2. Let $f : \{-1,1\}^n \rightarrow \{-1,1\}$. Let $x, x' \sim \{-1,1\}^n$ be independent uniformly random strings and let $\mu = \mathbb{E}[f(x)]$. Show that

$$\text{Var}[f] = \frac{1}{2} \mathbb{E}[(f(x) - f(x'))^2] = 4 \mathbb{Pr}[f(x) = 1] \mathbb{Pr}[f(x) = -1] = 2 \mathbb{Pr}[f(x) \neq f(x')] = \mathbb{E}[(f(x) - \mu)$.]

3. The (boolean) dual of $f : \{-1,1\}^n \rightarrow \mathbb{R}$ is the function $f^\dagger$ defined by $f^\dagger(x) = -f(-x)$. The function $f$ is said to be odd if it equals its dual; equivalently, if $f(-x) = -f(x)$ for all $x$. The function $f$ is said to be even if $f(-x) = f(x)$ for all $x$. Given any function $f : \{-1,1\}^n \rightarrow \mathbb{R}$, its odd part is the function $f^{\text{odd}} : \{-1,1\}^n \rightarrow \mathbb{R}$ defined by $f^{\text{odd}}(x) = (f(x) - f(-x))/2$, and its even part is the function $f^{\text{even}} : \{-1,1\}^n \rightarrow \mathbb{R}$ defined by $f^{\text{even}}(x) = (f(x) + f(-x))/2.$
4. Let \( A \) be a Hadamard matrix. A function of \( H \) is defined as the sum of the Walsh–Hadamard transforms of the functions on \( n \) variables. Examples are the Walsh–Hadamard matrices \( H_N \), inductively defined for \( N = 2^n \) as follows:

\[
H_1 = [1], \quad H_{2n+1} = \begin{bmatrix} H_{2n} & H_{2n} \\ H_{2n} & -H_{2n} \end{bmatrix}.
\]

(a) Let's index the rows and columns of \( H \) by the integers \( \{0, 1, 2, \ldots, 2^n - 1\} \) rather than \( [2^n] \). Further, let's identify such an integer \( i \) with its binary expansion \( (i_0, i_1, \ldots, i_{n-1}) \in \mathbb{F}_2^n \), where \( i_0 \) is the least significant bit and \( i_{n-1} \) the most. E.g., if \( n = 3 \), we identify the index \( i = 6 \) with \( (0, 1, 1) \). Now show that the \( (\gamma, x) \) entry of \( H_{2n} \) is \((-1)^{\gamma \cdot x}\).

(b) Show that if \( f : \mathbb{F}_2^n \to \mathbb{R} \) is represented as a column vector in \( \mathbb{R}^{2^n} \) (according to the indexing scheme from part (a)) then \( 2^{-n} H_{2n} f = \hat{f} \). Here we think of \( \hat{f} \) as also being a function \( \mathbb{F}_2^n \to \mathbb{R} \), identifying subsets \( S \subseteq \{0, 1, \ldots, n-1\} \) with their indicator vectors.

(c) Show that taking the Fourier transform is essentially an “involution”: \( \hat{\hat{f}} = 2^{-n} f \) (using the notations from part (b)).

(d) (Optional.) Show how to compute \( H_{2n} f \) using just \( n2^n \) additions and subtractions (rather than \( 2^{2n} \) additions and subtractions as the usual matrix-vector multiplication algorithm would require). This computation is called the Fast Walsh–Hadamard Transform and is the method of choice for computing the Fourier expansion of a generic function \( f : \mathbb{F}_2^n \to \mathbb{R} \) when \( n \) is large.

5. A Hadamard matrix is any \( N \times N \) real matrix with \pm 1 entries and orthogonal rows. Particular examples are the Walsh–Hadamard matrices \( H_N \), inductively defined for \( N = 2^n \) as follows:

\[
H_1 = [1], \quad H_{2n+1} = \begin{bmatrix} H_{2n} & H_{2n} \\ H_{2n} & -H_{2n} \end{bmatrix}.
\]

(a) Suppose \( W^1[f] = 1 \). Show that \( f(x) = \pm \chi_S \) for some \( |S| = 1 \).

(b) Suppose \( W^1_1[f] = 1 \). Show that \( f \) depends on at most 1 input coordinate.

(c) Suppose \( W^1_2[f] = 1 \). Is it true that \( f \) depends on at most 2 input coordinates?

6. (Sanders '06.) Let \( A \subseteq \mathbb{F}_2^n \), let \( \alpha = |A|/2^n \), and write \( 1_A : \mathbb{F}_2^n \to \{0, 1\} \) for the indicator function of \( A \).

(a) Show that \( \sum_{S \neq \emptyset} \hat{1}_A(S)^2 = \alpha (1 - \alpha) \).

(b) Define \( A + A + A = \{ x + y + z : x, y, z \in A \} \), where the addition is in \( \mathbb{F}_2^n \). Show that either \( A + A + A = \mathbb{F}_2^n \) or else there exists \( S^* \neq \emptyset \) such that \( |\hat{1}_A(S^*)| \geq \frac{1}{2n} \cdot \alpha \). (Hint: if \( A + A + A \neq \mathbb{F}_2^n \), show there exists \( x \in \mathbb{F}_2^n \) such that \( 1_A * 1_A * 1_A(x) = 0 \).)