Safe and Nested Subgame Solving for Imperfect-Information Games

Noam Brown
Computer Science Department
Carnegie Mellon University
Pittsburgh, PA 15217
noamb@cs.cmu.edu

Tuomas Sandholm
Computer Science Department
Carnegie Mellon University
Pittsburgh, PA 15217
sandholm@cs.cmu.edu

Abstract

In imperfect-information games, the optimal strategy in a subgame may depend on the strategy in other, unreached subgames. Thus a subgame cannot be solved in isolation and must instead consider the strategy for the entire game as a whole, unlike perfect-information games. Nevertheless, it is possible to first approximate a solution for the whole game and then improve it in individual subgames. This is referred to as *subgame solving*. We introduce subgame-solving techniques that outperform prior methods both in theory and practice. We also show how to adapt them, and past subgame-solving techniques, to respond to opponent actions that are outside the original action abstraction; this significantly outperforms the prior state-of-the-art approach, action translation. Finally, we show that subgame solving can be repeated as the game progresses down the game tree, leading to far lower exploitability. These techniques were a key component of *Libratus*, the first AI to defeat top humans in heads-up no-limit Texas hold'em poker.

1 Introduction

Imperfect-information games model strategic settings that have hidden information. They have a myriad of applications including negotiation, auctions, cybersecurity, and physical security.

In perfect-information games, determining the optimal strategy at a decision point only requires knowledge of the game tree's current node and the remaining game tree beyond that node (the *subgame* rooted at that node). This fact has been leveraged by nearly every AI for perfect-information games, including AIs that defeated top humans in chess [6] and Go [28]. In checkers, the ability to decompose the game into smaller independent subgames was even used to solve the entire game [26]. However, it is not possible to determine a subgame's optimal strategy in an imperfect-information game using only knowledge of that subgame, because the game tree's exact node is typically unknown. Instead, the optimal strategy may depend on the value an opponent could have received in some other, unreached subgame. Although this is counter-intuitive, we provide a demonstration in Section 2.

Rather than rely on subgame decomposition, past approaches for imperfect-information games typically solved the game as a whole upfront. For example, heads-up limit Texas hold'em, a relatively simple form of poker with 10^{13} decision points, was essentially solved without decomposition [2]. However, this approach cannot extend to larger games, such as heads-up no-limit Texas hold'em—the primary benchmark in imperfect-information game solving—which has 10^{161} decision points [15].

The standard approach to computing strategies in such large games is to first generate an *abstraction* of the game, which is a smaller version of the game that retains as much as possible the strategic characteristics of the original game [23, 25, 24]. For example, a continuous action space might be discretized. This abstract game is solved and its solution is used when playing the full game by mapping states in the full game to states in the abstract game. We refer to the solution of an abstraction (or more generally any approximate solution to a game) as a *blueprint* strategy.

31st Conference on Neural Information Processing Systems (NIPS 2017), Long Beach, CA, USA.

In heavily abstracted games, a blueprint strategy may be far from the true solution. *Subgame solving* attempts to improve upon the blueprint strategy by solving in real time a more fine-grained abstraction for an encountered subgame, while fitting its solution within the overarching blueprint strategy.

2 Coin Toss

In this section we provide intuition for why an imperfect-information subgame cannot be solved in isolation. We demonstrate this in a simple game we call Coin Toss, shown in Figure 1a, which will be used as a running example throughout the paper.

Coin Toss is played between players P_1 and P_2. The figure shows rewards only for P_1; P_2 always receives the negation of P_1's reward. A coin is flipped and lands either Heads or Tails with equal probability, but only P_1 sees the outcome. P_1 then chooses between actions "Sell" and "Play." The Sell action leads to a subgame whose details are not important, but the *expected value* (EV) of choosing the Sell action will be important. (For simplicity, one can equivalently assume *in this section* that Sell leads to an immediate terminal reward, where the value depends on whether the coin landed Heads or Tails). If the coin lands Heads, it is considered lucky and P_1 receives an EV of $\$0.50$ for choosing Sell. On the other hand, if the coin lands Tails, it is considered unlucky and P_1 receives an EV of $-\$0.50$ for action Sell. (That is, P_1 must on average pay $\$0.50$ to get rid of the coin). If P_1 instead chooses Play, then P_2 may guess how the coin landed. If P_2 guesses correctly, then P_1 receives a reward of $-\$1$. If P_2 guesses incorrectly, then P_1 receives $\$1$. P_2 may also forfeit, which should never be chosen but will be relevant in later sections. We wish to determine the optimal strategy for P_2 in the subgame S that occurs after P_1 chooses Play, shown in Figure 1a.

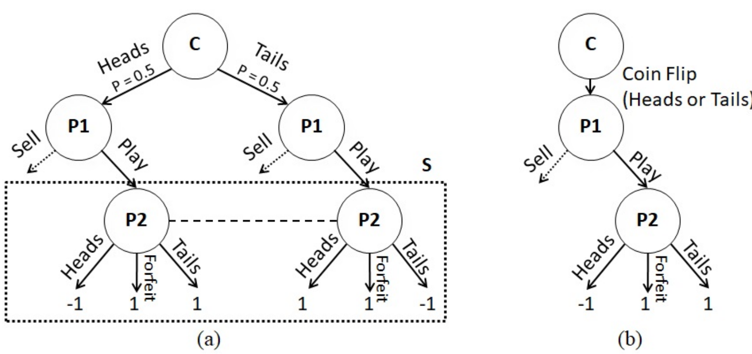


Figure 1: (a) The example game of Coin Toss. "C" represents a chance node. S is a Player 2 (P_2) subgame. The dotted line between the two P_2 nodes means that P_2 cannot distinguish between them. (b) The public game tree of Coin Toss. The two outcomes of the coin flip are only observed by P_1.

Were P_2 to always guess Heads, P_1 would receive $\$0.50$ for choosing Sell when the coin lands Heads, and $\$1$ for Play when it lands Tails. This would result in an average of $\$0.75$ for P_1. Alternatively, were P_2 to always guess Tails, P_1 would receive $\$1$ for choosing Play when the coin lands Heads, and $-\$0.50$ for choosing Sell when it lands Tails. This would result in an average reward of $\$0.25$ for P_1. However, P_2 would do even better by guessing Heads with 25% probability and Tails with 75% probability. In that case, P_1 could only receive $\$0.50$ (on average) by choosing Play when the coin lands Heads—the same value received for choosing Sell. Similarly, P_1 could only receive $-\$0.50$ by choosing Play when the coin lands Tails, which is the same value received for choosing Sell. This would yield an average reward of $\$0$ for P_1. It is easy to see that this is the best P_2 can do, because P_1 can average $\$0$ by always choosing Sell. Therefore, choosing Heads with 25% probability and Tails with 75% probability is an optimal strategy for P_2 in the "Play" subgame.

Now suppose the coin is considered lucky if it lands Tails and unlucky if it lands Heads. That is, the expected reward for selling the coin when it lands Heads is now $-\$0.50$ and when it lands Tails is now $\$0.50$. It is easy to see that P_2's optimal strategy for the "Play" subgame is now to guess Heads with 75% probability and Tails with 25% probability. This shows that a player's optimal strategy in a subgame can depend on the strategies and outcomes in other parts of the game. Thus, one cannot solve a subgame using information about that subgame alone. This is the central challenge of imperfect-information games as opposed to perfect-information games.

3 Notation and Background

In a two-player zero-sum extensive-form game there are two players, $\mathcal{P} = \{1, 2\}$. H is the set of all possible nodes, represented as a sequence of actions. $A(h)$ is the actions available in a node and $P(h) \in \mathcal{P} \cup c$ is the player who acts at that node, where c denotes chance. Chance plays an action $a \in A(h)$ with a fixed probability. If action $a \in A(h)$ leads from h to h', then we write $h \cdot a = h'$. If a sequence of actions leads from h to h', then we write $h \sqsubset h'$. The set of nodes $Z \subseteq H$ are terminal nodes. For each player $i \in \mathcal{P}$, there is a payoff function $u_i : Z \to \Re$ where $u_1 = -u_2$.

Imperfect information is represented by *information sets* (infosets). Every node $h \in H$ belongs to exactly one infoset for each player. For any infoset I_i, nodes $h, h' \in I_i$ are indistinguishable to player i. Thus the same player must act at all the nodes in an infoset, and the same actions must be available. Let $P(I_i)$ and $A(I_i)$ be such that all $h \in I_i$, $P(I_i) = P(h)$ and $A(I_i) = A(h)$.

A strategy $\sigma_i(I_i)$ is a probability vector over $A(I_i)$ for infosets where $P(I_i) = i$. The probability of action a is denoted by $\sigma_i(I_i, a)$. For all $h \in I_i$, $\sigma_i(h) = \sigma_i(I_i)$. A full-game strategy $\sigma_i \in \Sigma_i$ defines a strategy for each player i infoset. A strategy profile σ is a tuple of strategies, one for each player. The expected payoff for player i if all players play the strategy profile $\langle \sigma_i, \sigma_{-i} \rangle$ is $u_i(\sigma_i, \sigma_{-i})$, where σ_{-i} denotes the strategies in σ of all players other than i.

Let $\pi^\sigma(h) = \prod_{h' \cdot a \sqsubseteq h} \sigma_{P(h')}(h', a)$ denote the probability of reaching h if all players play according to σ. $\pi_i^\sigma(h)$ is the contribution of player i to this probability (that is, the probability of reaching h if chance and all players other than i always chose actions leading to h). $\pi_{-i}^\sigma(h)$ is the contribution of all players, and chance, *other than* i. $\pi^\sigma(h, h')$ is the probability of reaching h' given that h has been reached, and 0 if $h \not\sqsubset h'$. This papers focuses on *perfect-recall* games, where a player never forgets past information. Thus, for every I_i, $\forall h, h' \in I_i$, $\pi_i^\sigma(h) = \pi_i^\sigma(h')$. We define $\pi_i^\sigma(I_i) = \pi_i^\sigma(h)$ for $h \in I_i$. Also, $I'_i \sqsubset I_i$ if for some $h' \in I'_i$ and some $h \in I_i$, $h' \sqsubset h$. Similarly, $I'_i \cdot a \sqsubset I_i$ if $h' \cdot a \sqsubset h$.

A *Nash equilibrium* [21] is a strategy profile σ^* where no player can improve by shifting to a different strategy, so σ^* satisfies $\forall i,\ u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma'_i \in \Sigma_i} u_i(\sigma'_i, \sigma_{-i}^*)$. A *best response* $BR(\sigma_{-i})$ is a strategy for player i that is optimal against σ_{-i}. Formally, $BR(\sigma_{-i})$ satisfies $u_i(BR(\sigma_{-i}), \sigma_{-i}) = \max_{\sigma'_i \in \Sigma_i} u_i(\sigma'_i, \sigma_{-i})$. In a two-player zero-sum game, the *exploitability* *exp*(σ_i) of a strategy σ_i is how much worse σ_i does against an opponent best response than a Nash equilibrium strategy would do. Formally, exploitability of σ_i is $u_i(\sigma^*) - u_i(\sigma_i, BR(\sigma_i))$, where σ^* is a Nash equilibrium.

The expected *value* of a node h when players play according to σ is $v_i^\sigma(h) = \sum_{z \in Z} \big(\pi^\sigma(h, z) u_i(z)\big)$. An infoset's value is the weighted average of the values of the nodes in the infoset, where a node is weighed by the player's belief that she is in that node. Formally, $v_i^\sigma(I_i) = \frac{\sum_{h \in I_i} \left(\pi_{-i}^\sigma(h) v_i^\sigma(h)\right)}{\sum_{h \in I_i} \pi_{-i}^\sigma(h)}$ and $v_i^\sigma(I_i, a) = \frac{\sum_{h \in I_i} \left(\pi_{-i}^\sigma(h) v_i^\sigma(h \cdot a)\right)}{\sum_{h \in I_i} \pi_{-i}^\sigma(h)}$. A *counterfactual best response* [20] $CBR(\sigma_{-i})$ is a best response that also maximizes value in unreached infosets. Specifically, a counterfactual best response is a best response σ_i with the additional condition that if $\sigma_i(I_i, a) > 0$ then $v_i^\sigma(I_i, a) = \max_{a'} v_i^\sigma(I_i, a')$. We further define *counterfactual best response value* $CBV^{\sigma_{-i}}(I_i)$ as the value player i expects to achieve by playing according to $CBR(\sigma_{-i})$, having already reached infoset I_i. Formally, $CBV^{\sigma_{-i}}(I_i) = v_i^{\langle CBR(\sigma_{-i}), \sigma_{-i} \rangle}(I_i)$ and $CBV^{\sigma_{-i}}(I_i, a) = v_i^{\langle CBR(\sigma_{-i}), \sigma_{-i} \rangle}(I_i, a)$.

An *imperfect-information subgame*, which we refer to simply as a *subgame* in this paper, can in most cases (but not all) be described as including all nodes which share prior *public* actions (that is, actions viewable to both players). In poker, for example, a subgame is uniquely defined by a sequence of bets and public board cards. Figure 1b shows the public game tree of Coin Toss. Formally, an imperfect-information subgame is a set of nodes $S \subseteq H$ such that for all $h \in S$, if $h \sqsubset h'$, then $h' \in S$, and for all $h \in S$ and all $i \in \mathcal{P}$, if $h' \in I_i(h)$ then $h' \in S$. Define S_{top} as the set of earliest-reachable nodes in S. That is, $h \in S_{top}$ if $h \in S$ and $h' \notin S$ for any $h' \sqsubset h$.

4 Prior Approaches to Subgame Solving

This section reviews prior techniques for subgame solving in imperfect-information games, which we build upon. Throughout this section, we refer to the Coin Toss game shown in Figure 1a.

As discussed in Section 1, a standard approach to dealing with large imperfect-information games is to solve an abstraction of the game. The abstract solution is a (probably suboptimal) strategy profile in the full game. We refer to this full-game strategy profile as the blueprint. The goal of subgame solving is to improve upon the blueprint by changing the strategy only in a subgame.

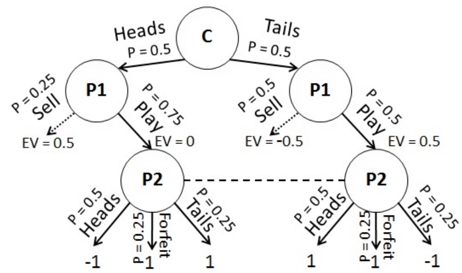


Figure 2: The blueprint strategy we refer to in the game of Coin Toss. The Sell action leads to a subgame that is not displayed. Probabilities are shown for all actions. The dotted line means the two P_2 nodes share an infoset. The EV of each P_1 action is also shown.

Assume that a blueprint strategy profile σ (shown in Figure 2) has already been computed for Coin Toss in which P_1 chooses Play $\frac{3}{4}$ of the time with Heads and $\frac{1}{2}$ of the time with Tails, and P_2 chooses Heads $\frac{1}{2}$ of the time, Tails $\frac{1}{4}$ of the time, and Forfeit $\frac{1}{4}$ of the time after P_1 chooses Play. The details of the blueprint strategy in the Sell subgame are not relevant in this section, but the EV for choosing the Sell action *is* relevant. We assume that if P_1 chose the Sell action and played optimally thereafter, then she would receive an expected payoff of 0.5 if the coin is Heads, and -0.5 if the coin is Tails. We will attempt to improve P_2's strategy in the subgame S that follows P_1 choosing Play.

4.1 Unsafe Subgame Solving

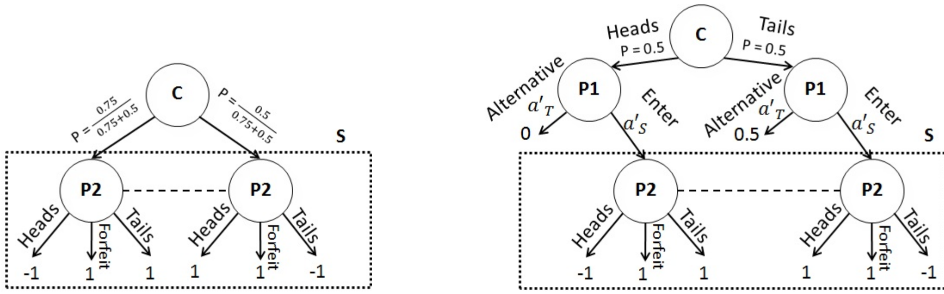
We first review the most intuitive form of subgame solving, which we refer to as *Unsafe subgame solving* [1, 11, 12, 9]. This form of subgame solving assumes both players played according to the blueprint strategy prior to reaching the subgame. That defines a probability distribution over the nodes at the root of the subgame S, representing the probability that the true game state matches that node. A strategy for the subgame is then calculated which assumes that this distribution is correct.

In all subgame solving algorithms, an *augmented subgame* containing S and a few additional nodes is solved to determine the strategy for S. Applying Unsafe subgame solving to the blueprint strategy in Coin Toss (after P_1 chooses Play) means solving the augmented subgame shown in Figure 3a.

Specifically, the augmented subgame consists of only an initial chance node and S. The initial chance node reaches $h \in S_{top}$ with probability $\frac{\pi^{\sigma}(h)}{\sum_{h' \in S_{top}} \pi^{\sigma}(h')}$. The augmented subgame is solved and its strategy for P_2 is used in S rather than the blueprint strategy.

Unsafe subgame solving lacks theoretical solution quality guarantees and there are many situations where it performs extremely poorly. Indeed, if it were applied to the blueprint strategy of Coin Toss then P_2 would always choose Heads—which P_1 could exploit severely by only choosing Play with Tails. Despite the lack of theoretical guarantees and potentially bad performance, Unsafe subgame solving is simple and can *sometimes* produce low-exploitability strategies, as we show later.

We now move to discussing *safe* subgame-solving techniques, that is, ones that ensure that the exploitability of the strategy is no higher than that of the blueprint strategy.



(a) Unsafe subgame solving

(b) Resolve subgame solving

Figure 3: The augmented subgames solved to find a P_2 strategy in the Play subgame of Coin Toss.

4.2 Subgame Resolving

In *subgame Resolving* [5], a safe strategy is computed for P_2 in the subgame by solving the augmented subgame shown in Figure 3b, producing an equilibrium strategy σ^S. This augmented subgame differs from Unsafe subgame solving by giving P_1 the option to "opt out" from entering S and instead receive the EV of playing optimally against P_2's blueprint strategy in S.

Specifically, the augmented subgame for Resolving differs from unsafe subgame solving as follows. For each $h_{top} \in S_{top}$ we insert a new P_1 node h_r, which exists only in the augmented subgame, between the initial chance node and h_{top}. The set of these h_r nodes is S_r. The initial chance node connects to each node $h_r \in S_r$ in proportion to the probability that player P_1 could reach h_{top} if P_1 tried to do so (that is, in proportion to $\pi^{\sigma}_{-1}(h_{top})$). At each node $h_r \in S_r$, P_1 has two possible actions. Action a'_S leads to h_{top}, while action a'_T leads to a terminal payoff that awards the value of playing optimally against P_2's blueprint strategy, which is $CBV^{\sigma_2}(I_1(h_{top}))$. In the blueprint strategy of Coin Toss, P_1 choosing Play after the coin lands Heads results in an EV of 0, and $\frac{1}{2}$ if the coin is Tails. Therefore, a'_T leads to a terminal payoff of 0 for Heads and $\frac{1}{2}$ for Tails. After the equilibrium strategy σ^S is computed in the augmented subgame, P_2 plays according to the computed subgame strategy σ^S_2 rather than the blueprint strategy when in S. The P_1 strategy σ^S_1 is not used.

Clearly P_1 cannot do worse than always picking action a'_T (which awards the highest EV P_1 could achieve against P_2's blueprint). But P_1 also cannot do *better* than always picking a'_T, because P_2 could simply play according to the blueprint in S, which means action a'_S would give the same EV to P_1 as action a'_T (if P_1 played optimally in S). In this way, the strategy for P_2 in S is pressured to be no worse than that of the blueprint. In Coin Toss, if P_2 were to always choose Heads (as was the case in Unsafe subgame solving), then P_1 would always choose a'_T with Heads and a'_S with Tails.

Resolving guarantees that P_2's exploitability will be no higher than the blueprint's (and may be better). However, it may miss opportunities for improvement. For example, if we apply Resolving to the example blueprint in Coin Toss, one solution to the augmented subgame is the blueprint itself, so P_2 may choose Forfeit 25% of the time even though Heads and Tails dominate that action. Indeed, the original purpose of Resolving was not to *improve* upon a blueprint strategy in a subgame, but rather to compactly store it by keeping only the EV at the root of the subgame and then reconstructing the strategy in real time when needed rather than storing the whole subgame strategy.

Maxmargin subgame solving [20], discussed in Appendix A, can improve performance by defining a *margin* $M^{\sigma^S}(I_1) = CBV^{\sigma_2}(I_1) - CBV^{\sigma^S_2}(I_1)$ for each $I_1 \in S_{top}$ and maximizing $\min_{I_1 \in S_{top}} M^{\sigma^S}(I_1)$. Resolving only makes all margins nonnegative. However, Maxmargin does worse in practice when using estimates of equilibrium values as discussed in Appendix C.

5 Reach Subgame Solving

All of the subgame-solving techniques described in Section 4 only consider the target subgame in isolation, which can lead to suboptimal strategies. For example, Maxmargin solving applied to S in Coin Toss results in P_2 choosing Heads with probability $\frac{5}{8}$ and Tails with $\frac{3}{8}$ in S. This results in P_1 receiving an EV of $-\frac{1}{4}$ by choosing Play in the Heads state, and an EV of $\frac{1}{4}$ in the Tails state. However, P_1 could simply always choose Sell in the Heads state (earning an EV of 0.5) and Play in the Tails state and receive an EV of $\frac{3}{8}$ for the entire game. In this section we introduce *Reach subgame solving*, an improvement to past subgame-solving techniques that considers *what the opponent could have alternatively received from other subgames*.[1] For example, a better strategy for P_2 would be to choose Heads with probability $\frac{3}{4}$ and Tails with probability $\frac{1}{4}$. Then P_1 is indifferent between choosing Sell and Play in both cases and overall receives an expected payoff of 0 for the whole game.

However, that strategy is only optimal if P_1 would indeed achieve an EV of 0.5 for choosing Sell in the Heads state and -0.5 in the Tails state. That would be the case if P_2 played according to the blueprint in the Sell subgame (which is not shown), but in reality we would apply subgame solving to the Sell subgame if the Sell action were taken, which would change P_2's strategy there and therefore P_1's EVs. Applying subgame solving to any subgame encountered during play is equivalent to applying it to all subgames independently; ultimately, the same strategy is played in both cases. Thus, we must consider that the EVs from other subgames may differ from what the blueprint says because subgame solving would be applied to them as well.

[1]Other subgame-solving methods have also considered the cost of reaching a subgame [30, 14]. However, those approaches are not correct in theory when applied in real time to any subgame reached during play.

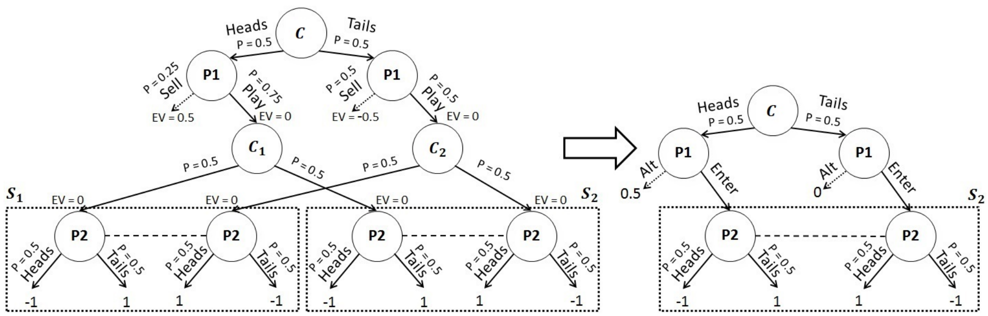


Figure 4: Left: A modified game of Coin Toss with two subgames. The nodes C_1 and C_2 are public chance nodes whose outcomes are seen by both P_1 and P_2. Right: An augmented subgame for one of the subgames according to Reach subgame solving. If only one of the subgames is being solved, then the alternative payoff for Heads can be at most 1. However, if both are solved independently, then the gift must be split among the subgames and must sum to at most 1. For example, the alternative payoff in both subgames can be 0.5.

As an example of this issue, consider the game shown in Figure 4 which contains two identical subgames S_1 and S_2 where the blueprint has P_2 pick Heads and Tails with 50% probability. The Sell action leads to an EV of 0.5 from the Heads state, while Play leads to an EV of 0. If we were to solve just S_1, then P_2 could afford to always choose Tails in S_1, thereby letting P_1 achieve an EV of 1 for reaching that subgame from Heads because, due to the chance node C_1, S_1 is only reached with 50% probability. Thus, P_1's EV for choosing Play would be 0.5 from Heads and -0.5 from Tails, which is optimal. We can achieve this strategy in S_1 by solving an augmented subgame in which the alternative payoff for Heads is 1. In that augmented subgame, P_2 always choosing Tails would be a solution (though not the only solution).

However, if the same reasoning were applied independently to S_2 as well, then P_2 might always choose Tails in both subgames and P_1's EV for choosing Play from Heads would become 1 while the EV for Sell would only be 0.5. Instead, we could allow P_1 to achieve an EV of 0.5 for reaching each subgame from Heads (by setting the alternative payoff for Heads to 0.5). In that case, P_1's overall EV for choosing Play could only increase to 0.5, even if both S_1 and S_2 were solved independently.

We capture this intuition by considering for each $I_1 \in S_{top}$ all the infosets and actions $I_1' \cdot a' \sqsubset I_1$ that P_1 would have taken along the path to I_1. If, at some $I_1' \cdot a' \sqsubset I_1$ where P_1 acted, there was a different action $a^* \in A(I_1')$ that leads to a higher EV, then P_1 would have taken a suboptimal action if they reached I_1. The difference in value between a^* and a' is referred to as a *gift*. We can afford to let P_1's value for I_1 increase beyond the blueprint value (and in the process lower P_1's value in some other infoset in S_{top}), so long as the increase to I_1's value is small enough that choosing actions leading to I_1 is still suboptimal for P_1. Critically, we must ensure that the increase in value is small enough even when the potential increase across all subgames is summed together, as in Figure 4.[2]

A complicating factor is that gifts we assumed were present may actually not exist. For example, in Coin Toss, suppose applying subgame solving to the Sell subgame results in P_1's value for Sell from the Heads state decreasing from 0.5 to 0.25. If we independently solve the Play subgame, we have no way of knowing that P_1's value for Sell is lower than the blueprint suggested, so we may still assume there is a gift of 0.5 from the Heads state based on the blueprint. Thus, in order to guarantee a theoretical result on exploitability that is as strong as possible, we use in our theory and experiments a *lower bound* on what gifts could be after subgame solving was applied to all other subgames.

Formally, let σ_2 be a P_2 blueprint and let σ_2^{-S} be the P_2 strategy that results from applying subgame solving independently to a set of disjoint subgames other than S. Since we do not want to compute σ_2^{-S} in order to apply subgame solving to S, let $\lfloor g^{\sigma_2^{-S}}(I_1', a') \rfloor$ be a lower bound of $CBV^{\sigma_2^{-S}}(I_1') - CBV^{\sigma_2^{-S}}(I_1', a')$ that does not require knowledge of σ_2^{-S}. In our experiments we

[2] In this paper and in our experiments, we allow any infoset that descends from a gift to increase by the size of the gift (e.g., in Figure 4 the gift from Heads is 0.5, so we allow P_1's value for Heads in both S_1 and S_2 to increase by 0.5). However, any division of the gift among subgames is acceptable so long as the potential increase across all subgames (multiplied by the probability of P_1 reaching that subgame) does not exceed the original gift. For example in Figure 4 if we only apply Reach subgame solving to S_1, then we could allow the Heads state in S_1 to increase by 1 rather than just by 0.5. In practice, some divisions may do better than others. The division we use in this paper (applying gifts equally to all subgames) did well in practice.

use $\lfloor g^{\sigma_2^{-S}}(I'_1, a') \rfloor = \max_{a \in A_z(I'_1) \cup \{a'\}} CBV^{\sigma_2}(I'_1, a) - CBV^{\sigma_2}(I'_1, a')$ where $A_z(I'_1) \subseteq A(I'_1)$ is the set of actions leading immediately to terminal nodes. Reach subgame solving modifies the augmented subgame in Resolving and Maxmargin by increasing the alternative payoff for infoset $I_1 \in S_{top}$ by $\sum_{I'_1 \cdot a' \sqsubseteq I_1 | P(I'_1) = P_1} \lfloor g^{\sigma_2^{-S}}(I'_1, a') \rfloor$. Formally, we define a *reach margin* as

$$M_r^{\sigma^S}(I_1) = M^{\sigma^S}(I_1) + \sum_{I'_1 \cdot a' \sqsubseteq I_1 | P(I'_1) = P_1} \lfloor g^{\sigma_2^{-S}}(I'_1, a') \rfloor \quad (1)$$

This margin is larger than or equal to the one for Maxmargin, because $\lfloor g^{\sigma_2^{-S}}(I', a') \rfloor$ is nonnegative. We refer to the modified algorithms as Reach-Resolve and Reach-Maxmargin.

Using a lower bound on gifts is not necessary to guarantee safety. So long as we use a gift value $g^{\sigma'}(I'_1, a') \leq CBV^{\sigma_2}(I'_1) - CBV^{\sigma_2}(I'_1, a')$, the resulting strategy will be safe. However, using a lower bound further guarantees a reduction to exploitability when a P_1 best response reaches with positive probability an infoset $I_1 \in S_{top}$ that has positive margin, as proven in Theorem 1. In practice, it may be best to use an accurate estimate of gifts. One option is to use $\hat{g}^{\sigma_2^{-S}}(I'_1, a') = C\tilde{B}V^{\sigma_2}(I'_1) - C\tilde{B}V^{\sigma_2}(I'_1, a')$ in place of $\lfloor g^{\sigma_2^{-S}}(I'_1, a') \rfloor$, where $C\tilde{B}V^{\sigma_2}$ is the closest P_1 can get to the value of a counterfactual best response while P_1 is constrained to playing within the abstraction that generated the blueprint. Using estimates is covered in more detail in Appendix C.

Theorem 1 shows that when subgames are solved independently and using lower bounds on gifts, Reach-Maxmargin solving has exploitability lower than or equal to past safe techniques. The theorem statement is similar to that of Maxmargin [20], but the margins are now larger (or equal) in size.

Theorem 1. *Given a strategy σ_2 in a two-player zero-sum game, a set of disjoint subgames $\mathbb{S}$, and a strategy σ_2^S for each subgame $S \in \mathbb{S}$ produced via Reach-Maxmargin solving using lower bounds for gifts, let σ'_2 be the strategy that plays according to σ_2^S for each subgame $S \in \mathbb{S}$, and σ_2 elsewhere. Moreover, let σ_2^{-S} be the strategy that plays according to σ'_2 everywhere except for P_2 nodes in S, where it instead plays according to σ_2. If $\pi_1^{BR(\sigma'_2)}(I_1) > 0$ for some $I_1 \in S_{top}$, then $exp(\sigma'_2) \leq exp(\sigma_2^{-S}) - \sum_{h \in I_1} \pi_{-1}^{\sigma_2}(h) M_r^{\sigma^S}(I_1)$.*

So far the described techniques have guaranteed a reduction in exploitability over the blueprint by setting the value of a'_T equal to the value of P_1 playing optimally to P_2's blueprint. Relaxing this guarantee by instead setting the value of a'_T equal to an *estimate* of P_1's value when *both* players play optimally leads to far lower exploitability in practice. We discuss this approach in Appendix C.

6 Nested Subgame Solving

As we have discussed, large games must be abstracted to reduce the game to a tractable size. This is particularly common in games with large or continuous action spaces. Typically the action space is discretized by action abstraction so that only a few actions are included in the abstraction. While we might limit ourselves to the actions we included in the abstraction, an opponent might choose actions that are not in the abstraction. In that case, the *off-tree* action can be mapped to an action that is in the abstraction, and the strategy from that in-abstraction action can be used. For example, in an auction game we might include a bid of \$100 in our abstraction. If a player bids \$101, we simply treat that as a bid of \$100. This is referred to as *action translation* [13, 27, 7]. Action translation is the state-of-the-art prior approach to dealing with this issue. It has been used, for example, by all the leading competitors in the Annual Computer Poker Competition (ACPC).

In this section, we develop techniques for applying subgame solving to calculate responses to opponent off-tree actions, thereby obviating the need for action translation. That is, rather than simply treat a bid of \$101 as \$100, we calculate in real time a unique response to the bid of \$101. This can also be done in a nested fashion in response to subsequent opponent off-tree actions. Additionally, these techniques can be used to solve finer-grained models as play progresses down the game tree.

We refer to the first method as the *inexpensive* method.[3] When P_1 chooses an off-tree action a, a subgame S is generated following that action such that for any infoset I_1 that P_1 might be in, $I_1 \cdot a \in S_{top}$. This subgame may itself be an abstraction. A solution σ^S is computed via subgame solving, and σ^S is combined with σ to form a new blueprint σ' in the expanded abstraction that now includes action a. The process repeats whenever P_1 again chooses an off-tree action.

[3]Following our study, the AI DeepStack used a technique similar to this form of nested subgame solving [19].

To conduct safe subgame solving in response to off-tree action a, we could calculate $CBV^{\sigma_2}(I_1, a)$ by defining, via action translation, a P_2 blueprint following a and best responding to it [4]. However, that could be computationally expensive and would likely perform poorly in practice because, as we show later, action translation is highly exploitable. Instead, we relax the guarantee of safety and use $C\tilde{B}V^{\sigma_2}(I_1)$ for the alternative payoff, where $C\tilde{B}V^{\sigma_2}(I_1)$ is P_1's counterfactual best response value in I_1 when constrained to playing in the blueprint abstraction (which excludes action a). In this case, exploitability depends on how well $C\tilde{B}V^{\sigma_2}(I_1)$ approximates $CBV^{\sigma_2^*}(I_1)$, where σ_2^* is an optimal P_2 strategy (see Appendix C).[4] In general, we find that only a small number of near-optimal actions need to be included in the blueprint abstraction for $C\tilde{B}V^{\sigma_2}(I_1)$ to be close to $CBV^{\sigma_2^*}(I_1)$. We can then approximate a near-optimal response to any opponent action, even in a continuous action space.

The "inexpensive" approach cannot be combined with Unsafe subgame solving because the probability of reaching an action outside of a player's abstraction is undefined. Nevertheless, a similar approach is possible with Unsafe subgame solving (as well as all the other subgame-solving techniques) by starting the subgame solving at h rather than at $h \cdot a$. In other words, if action a taken in node h is not in the abstraction, then Unsafe subgame solving is conducted in the smallest subgame containing h (and action a is added to that abstraction). This increases the size of the subgame compared to the inexpensive method because a strategy must be recomputed for every action $a' \in A(h)$ in addition to a. We therefore call this method the *expensive* method. We present experiments with both methods.

7 Experiments

Our experiments were conducted on heads-up no-limit Texas hold'em, as well as two smaller-scale poker games we call *No-Limit Flop Hold'em* (NLFH) and *No-Limit Turn Hold'em* (NLTH). The description for these games can be found in Appendix G. For equilibrium finding, we used CFR+ [29].

Our first experiment compares the performance of the subgame-solving techniques when applied to information abstraction (which is card abstraction in the case of poker). Specifically, we solve NLFH with no information abstraction on the preflop. On the flop, there are 1,286,792 infosets for each betting sequence; the abstraction buckets them into 200, 2,000, or 30,000 abstract ones (using a leading information abstraction algorithm [8]). We then apply subgame solving immediately after the flop community cards are dealt. We experiment with two versions of the game, one small and one large, which include only a few of the available actions in each infoset. We also experimented on abstractions of NLTH. In that case, we solve NLTH with no information abstraction on the preflop or flop. On the turn, there are 55,190,538 infosets for each betting sequence; the abstraction buckets them into 200, 2,000, or 20,000 abstract ones. We apply subgame solving immediately after the turn community card is dealt. Table 1 shows the performance of each technique when using 30,000 buckets (20,000 for NLTH). The full results are presented in Appendix E. In all our experiments, exploitability is measured in the standard units used in this field: milli big blinds per hand (mbb/h).

	Small Flop Holdem	**Large Flop Holdem**	**Turn Holdem**
Blueprint Strategy	91.28	41.41	345.5
Unsafe	5.514	396.8	79.34
Resolve	54.07	23.11	251.8
Maxmargin	43.43	19.50	234.4
Reach-Maxmargin	41.47	18.80	233.5
Reach-Maxmargin (no split)	25.88	16.41	175.5
Estimate	24.23	30.09	76.44
Estimate+Distributional	34.30	10.54	74.35
Reach-Estimate+Distributional	22.58	9.840	72.59
Reach-Estimate+Distributional (no split)	17.33	8.777	70.68

Table 1: Exploitability of various subgame-solving techniques in three different games.

Estimate and Estimate+Distributional are techniques introduced in Appendix C. We use a normal distribution in the Distributional subgame solving experiments, with standard deviation determined by the heuristic presented in Appendix C.1.

Since subgame solving begins immediately after a chance node with an extremely high branching factor ($1,755$ in NLFH), the gifts for the Reach algorithms are divided among subgames inefficiently.

[4]We estimate $CBV^{\sigma_2^*}(I_1)$ rather than $CBV^{\sigma_2^*}(I_1, a)$ because $CBV^{\sigma_2^*}(I_1) - CBV^{\sigma_2^*}(I_1, a)$ is a gift that may be added to the alternative payoff anyway.

Many subgames do not use the gifts at all, while others could make use of more. In the experiments we show results both for the theoretically safe splitting of gifts, as well as a more aggressive version where gifts are scaled up by the branching factor of the chance node ($1,755$). This weakens the theoretical guarantees of the algorithm, but in general did better than splitting gifts in a theoretically correct manner. However, this is not universally true. Appendix F shows that in at least one case, exploitability increased when gifts were scaled up too aggressively. In all cases, using Reach subgame solving in at least the theoretical safe method led to lower exploitability.

Despite lacking theoretical guarantees, Unsafe subgame solving did surprisingly well in most games. However, it did substantially worse in Large NLFH with 30,000 buckets. This exemplifies its variability. Among the safe methods, all of the changes we introduce show improvement over past techniques. The Reach-Estimate + Distributional algorithm generally resulted in the lowest exploitability among the various choices, and in most cases beat unsafe subgame solving.

The second experiment evaluates nested subgame solving, and compares it to action translation. In order to also evaluate action translation, in this experiment, we create an NLFH game that includes 3 bet sizes at every point in the game tree (0.5, 0.75, and 1.0 times the size of the pot); a player can also decide not to bet. Only one bet (i.e., no raises) is allowed on the preflop, and three bets are allowed on the flop. There is no information abstraction anywhere in the game. We also created a second, smaller abstraction of the game in which there is still no information abstraction, but the $0.75\times$ pot bet is never available. We calculate the exploitability of one player using the smaller abstraction, while the other player uses the larger abstraction. Whenever the large-abstraction player chooses a $0.75\times$ pot bet, the small-abstraction player generates and solves a subgame for the remainder of the game (which again does not include any subsequent $0.75\times$ pot bets) using the nested subgame-solving techniques described above. This subgame strategy is then used as long as the large-abstraction player plays within the small abstraction, but if she chooses the $0.75\times$ pot bet again later, then the subgame solving is used again, and so on.

Table 2 shows that all the subgame-solving techniques substantially outperform action translation. We did not test distributional alternative payoffs in this experiment, since the calculated best response values are likely quite accurate. These results suggest that nested subgame solving is preferable to action translation (if there is sufficient time to solve the subgame).

	mbb/h
Randomized Pseudo-Harmonic Mapping	1,465
Resolve	150.2
Reach-Maxmargin (Expensive)	149.2
Unsafe (Expensive)	148.3
Maxmargin	122.0
Reach-Maxmargin	119.1

Table 2: Exploitability of the various subgame-solving techniques in nested subgame solving. The performance of the pseudo-harmonic action translation is also shown.

We used the techniques presented in this paper to develop *Libratus*, an AI that competed against four top human specialists in heads-up no-limit Texas hold'em. Heads-up no-limit Texas hold'em has been the primary benchmark challenge for AI in imperfect-information games. The competition involved 120,000 hands of poker and a prize pool of $200,000 split among the humans to incentivize strong play. The AI decisively defeated the human team by 147 mbb / hand, with 99.98% statistical significance. This was the first, and so far only, time an AI defeated top humans in no-limit poker.

8 Conclusion

We introduced a subgame-solving technique for imperfect-information games that has stronger theoretical guarantees and better practical performance than prior subgame-solving methods. We presented results on exploitability of both safe and unsafe subgame-solving techniques. We also introduced a method for nested subgame solving in response to the opponent's off-tree actions, and demonstrated that this leads to dramatically better performance than the usual approach of action translation. This is, to our knowledge, the first time that exploitability of subgame-solving techniques has been measured in large games.

Finally, we demonstrated the effectiveness of these techniques in practice in heads-up no-limit Texas Hold'em poker, the main benchmark challenge for AI in imperfect-information games. We developed the first AI to reach the milestone of defeating top humans in heads-up no-limit Texas Hold'em.

9 Acknowledgments

This material is based on work supported by the National Science Foundation under grants IIS-1718457, IIS-1617590, and CCF-1733556, and the ARO under award W911NF-17-1-0082, as well as XSEDE computing resources provided by the Pittsburgh Supercomputing Center. The *Brains vs. AI* competition was sponsored by Carnegie Mellon University, Rivers Casino, GreatPoint Ventures, Avenue4Analytics, TNG Technology Consulting, Artificial Intelligence, Intel, and Optimized Markets, Inc. We thank Kristen Gardner, Marcelo Gutierrez, Theo Gutman-Solo, Eric Jackson, Christian Kroer, Tim Reiff, and the anonymous reviewers for helpful feedback.

References

[1] Darse Billings, Neil Burch, Aaron Davidson, Robert Holte, Jonathan Schaeffer, Terence Schauenberg, and Duane Szafron. Approximating game-theoretic optimal strategies for full-scale poker. In *Proceedings of the 18th International Joint Conference on Artificial Intelligence (IJCAI)*, 2003.

[2] Michael Bowling, Neil Burch, Michael Johanson, and Oskari Tammelin. Heads-up limit hold'em poker is solved. *Science*, 347(6218):145–149, January 2015.

[3] Noam Brown, Christian Kroer, and Tuomas Sandholm. Dynamic thresholding and pruning for regret minimization. In *AAAI Conference on Artificial Intelligence (AAAI)*, pages 421–429, 2017.

[4] Noam Brown and Tuomas Sandholm. Simultaneous abstraction and equilibrium finding in games. In *Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI)*, 2015.

[5] Neil Burch, Michael Johanson, and Michael Bowling. Solving imperfect information games using decomposition. In *AAAI Conference on Artificial Intelligence (AAAI)*, pages 602–608, 2014.

[6] Murray Campbell, A Joseph Hoane, and Feng-Hsiung Hsu. Deep Blue. *Artificial intelligence*, 134(1-2):57–83, 2002.

[7] Sam Ganzfried and Tuomas Sandholm. Action translation in extensive-form games with large action spaces: axioms, paradoxes, and the pseudo-harmonic mapping. In *Proceedings of the Twenty-Third international joint conference on Artificial Intelligence*, pages 120–128. AAAI Press, 2013.

[8] Sam Ganzfried and Tuomas Sandholm. Potential-aware imperfect-recall abstraction with earth mover's distance in imperfect-information games. In *AAAI Conference on Artificial Intelligence (AAAI)*, 2014.

[9] Sam Ganzfried and Tuomas Sandholm. Endgame solving in large imperfect-information games. In *International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 37–45, 2015.

[10] Andrew Gilpin, Javier Peña, and Tuomas Sandholm. First-order algorithm with $\mathcal{O}(\ln(1/\epsilon))$ convergence for ϵ-equilibrium in two-person zero-sum games. *Mathematical Programming*, 133(1–2):279–298, 2012. Conference version appeared in AAAI-08.

[11] Andrew Gilpin and Tuomas Sandholm. A competitive Texas Hold'em poker player via automated abstraction and real-time equilibrium computation. In *Proceedings of the National Conference on Artificial Intelligence (AAAI)*, pages 1007–1013, 2006.

[12] Andrew Gilpin and Tuomas Sandholm. Better automated abstraction techniques for imperfect information games, with application to Texas Hold'em poker. In *International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 1168–1175, 2007.

[13] Andrew Gilpin, Tuomas Sandholm, and Troels Bjerre Sørensen. A heads-up no-limit texas hold'em poker player: discretized betting models and automatically generated equilibrium-finding programs. In *Proceedings of the Seventh International Joint Conference on Autonomous Agents and Multiagent Systems-Volume 2*, pages 911–918. International Foundation for Autonomous Agents and Multiagent Systems, 2008.

[14] Eric Jackson. A time and space efficient algorithm for approximately solving large imperfect information games. In *AAAI Workshop on Computer Poker and Imperfect Information*, 2014.

[15] Michael Johanson. Measuring the size of large no-limit poker games. Technical report, University of Alberta, 2013.

[16] Michael Johanson, Nolan Bard, Neil Burch, and Michael Bowling. Finding optimal abstract strategies in extensive-form games. In *Proceedings of the Twenty-Sixth AAAI Conference on Artificial Intelligence*, pages 1371–1379. AAAI Press, 2012.

[17] Christian Kroer, Kevin Waugh, Fatma Kılınç-Karzan, and Tuomas Sandholm. Theoretical and practical advances on smoothing for extensive-form games. In *Proceedings of the ACM Conference on Economics and Computation (EC)*, 2017.

[18] Nick Littlestone and M. K. Warmuth. The weighted majority algorithm. *Information and Computation*, 108(2):212–261, 1994.

[19] Matej Moravčík, Martin Schmid, Neil Burch, Viliam Lisý, Dustin Morrill, Nolan Bard, Trevor Davis, Kevin Waugh, Michael Johanson, and Michael Bowling. Deepstack: Expert-level artificial intelligence in heads-up no-limit poker. *Science*, 2017.

[20] Matej Moravcik, Martin Schmid, Karel Ha, Milan Hladik, and Stephen Gaukrodger. Refining subgames in large imperfect information games. In *AAAI Conference on Artificial Intelligence (AAAI)*, 2016.

[21] John Nash. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 36:48–49, 1950.

[22] Yurii Nesterov. Excessive gap technique in nonsmooth convex minimization. *SIAM Journal of Optimization*, 16(1):235–249, 2005.

[23] Tuomas Sandholm. The state of solving large incomplete-information games, and application to poker. *AI Magazine*, pages 13–32, Winter 2010. Special issue on Algorithmic Game Theory.

[24] Tuomas Sandholm. Abstraction for solving large incomplete-information games. In *AAAI Conference on Artificial Intelligence (AAAI)*, pages 4127–4131, 2015. Senior Member Track.

[25] Tuomas Sandholm. Solving imperfect-information games. *Science*, 347(6218):122–123, 2015.

[26] Jonathan Schaeffer, Neil Burch, Yngvi Björnsson, Akihiro Kishimoto, Martin Müller, Robert Lake, Paul Lu, and Steve Sutphen. Checkers is solved. *Science*, 317(5844):1518–1522, 2007.

[27] David Schnizlein, Michael Bowling, and Duane Szafron. Probabilistic state translation in extensive games with large action sets. In *Proceedings of the Twenty-First International Joint Conference on Artificial Intelligence*, pages 278–284, 2009.

[28] David Silver, Aja Huang, Chris J Maddison, Arthur Guez, Laurent Sifre, George Van Den Driessche, Julian Schrittwieser, Ioannis Antonoglou, Veda Panneershelvam, Marc Lanctot, et al. Mastering the game of go with deep neural networks and tree search. *Nature*, 529(7587):484–489, 2016.

[29] Oskari Tammelin, Neil Burch, Michael Johanson, and Michael Bowling. Solving heads-up limit texas hold'em. In *Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI)*, pages 645–652, 2015.

[30] Kevin Waugh, Nolan Bard, and Michael Bowling. Strategy grafting in extensive games. In *Proceedings of the Annual Conference on Neural Information Processing Systems (NIPS)*, 2009.

[31] Martin Zinkevich, Michael Johanson, Michael H Bowling, and Carmelo Piccione. Regret minimization in games with incomplete information. In *Proceedings of the Annual Conference on Neural Information Processing Systems (NIPS)*, pages 1729–1736, 2007.

Appendix: Supplementary Material

A Maxmargin Solving

Maxmargin solving [20] is similar to Resolving, except that it seeks to improve P_2's strategy in the subgame strategy as much as possible. While Resolving seeks a strategy for P_2 in S that would simply dissuade P_1 from entering S, Maxmargin solving additionally seeks to punish P_1 as much as possible if P_1 nevertheless chooses to enter S. A *subgame margin* is defined for each infoset in S_r, which represents the difference in value between entering the subgame versus choosing the alternative payoff. Specifically, for each infoset $I_1 \in S_{top}$, the *subgame margin* is

$$M^{\sigma^S}(I_1) = CBV^{\sigma_2}(I_1) - CBV^{\sigma_2^S}(I_1) \tag{2}$$

In Maxmargin solving, a Nash equilibrium σ^S for the augmented subgame described in Resolving subgame solving is computed such that the minimum margin over all $I_1 \in S_{top}$ is maximized. Aside from maximizing the minimum margin, the augmented subgames used in Resolving and Maxmargin solving are identical.

Given our base strategy in Coin Toss, Maxmargin solving would result in P_2 choosing Heads with probability $\frac{5}{8}$, Tails with probability $\frac{3}{8}$, and Forfeit with probability 0.

The augmented subgame can be solved in a way that maximizes the minimum margin by using a standard LP solver. In order to use iterative algorithms such as the Excessive Gap Technique [22, 10, 17] or Counterfactual Regret Minimization (CFR) [31], one can use the *gadget game* described by Moravcik et al. [20]. Details on the gadget game are provided in the Appendix. Our experiments used CFR.

Maxmargin solving is safe. Furthermore, it guarantees that if every Player 1 best response reaches the subgame with positive probability through some infoset(s) that have positive margin, then exploitability is strictly lower than that of the blueprint strategy. While the theoretical guarantees are stronger, Maxmargin may lead to worse practical performance relative to Resolving when combined with the techniques discussed in Appendix C, due to Maxmargin's greater tendency to overfit to assumptions in the model.

B Description of Gadget Game

Solving the augmented subgame described in Maxmargin solving and Reach-Maxmargin solving will not, by itself, necessarily maximize the minimum margin. While LP solvers can easily handle this objective, the process is more difficult for iterative algorithms such as Counterfactual Regret Minimization (CFR) and the Excessive Gap Technique (EGT). For these iterative algorithms, the augmented subgame can be modified into a *gadget game* that, when solved, will provide a Nash equilibrium to the augmented subgame and will also maximize the minimum margin [20]. This gadget game is unnecessary when using distributional alternative payoffs, which is introduced in section C.1.

The gadget game differs from the augmented subgame in two ways. First, all P_1 payoffs that are reached from the initial infoset of $I_1 \in S_r$ are shifted by the alternative payoff of I_1, and there is longer an alternative payoff. Second, rather than the game starting with a chance node that determines P_1's starting infoset, P_1 decides for herself which infoset to begin the game in. Specifically, the game begins with a P_1 node where each action in the node corresponds to an infoset I_1 in S_r. After P_1 chooses to enter an infoset I_1, chance chooses the precise node $h \in I_1$ in proportion to $\pi^{\sigma}_{-1}(h)$.

By shifting all payoffs in the game by the size of the alternative payoff, the gadget game forces P_1 to focus on improving the performance of each infoset over some baseline, which is the goal of Maxmargin and Reach-Maxmargin solving. Moreover, by allowing P_1 to choose the infoset in which to enter the game, the gadget game forces P_2 to focus on maximizing the minimum margin.

Figure 5 illustrates the gadget game used in Maxmargin and Reach-Maxmargin.

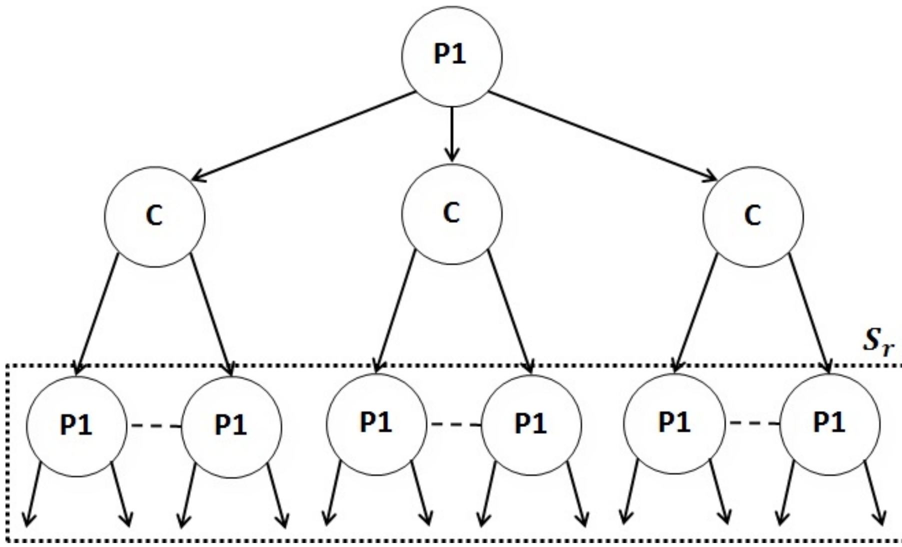


Figure 5: An example of a gadget game in Maxmargin refinement. P_1 picks the initial infoset she wishes to enter S_r in. Chance then picks the particular node of the infoset, and play then proceeds identically to the augmented subgame, except all P_1 payoffs are shifted by the size of the alternative payoff and the alternative payoff is then removed from the augmented subgame.

C Modeling Error in a Subgame

In this section we consider the case where we have a good estimate of what the values of subgames would look like in a Nash equilibrium. Unlike previous sections, exploitability might be *higher* than the blueprint when using this method; the solution quality ultimately depends on the accuracy of the estimates used. In practice this approach leads to significantly lower exploitability.

When solving multiple P_2 subgames, there is a minimally-exploitable strategy σ_2^* that could, in theory, be computed by changing only the strategies in the subgames. (σ_2^* may not be a Nash equilibrium because P_2's strategy outside the subgames is fixed, but it is the closest that can be achieved by changing the strategy only in the subgames). However, σ_2^* can only be guaranteed to be produced by solving all the subgames together, because the optimal strategy in one subgame depends on the optimal strategy in other subgames.

Still, suppose that we know $CBV^{\sigma_2^*}(I_1)$ for every infoset $I_1 \in S_{top}$ for every subgame S. Let $I_{r,1}$ be the infoset in S_r that leads to I_1. By setting the P_1 alternative payoff for $I_{r,1}$ to $v(I_{r,1}, a'_T) = CBV^{\sigma_2^*}(I_1)$, safe subgame solving guarantees a strategy will be produced with exploitability no worse than σ_2^*. Thus, achieving a strategy equivalent to σ_2^* does not require knowledge of σ_2^*; rather, it only requires knowledge of $CBV^{\sigma_2^*}(I_1)$ for infosets I_1 in the top of the subgames.

While we do not know $CBV^{\sigma_2^*}(I_1)$ exactly without knowing σ_2^* itself, we may nevertheless be able to produce (or learn) good *estimates* of $CBV^{\sigma_2^*}(I_1)$. For example, in Section 7 we compute the solution to the game of No-Limit Flop Hold'em (NLFH), and find that in perfect play P_2 can expect to win about 37 mbb/h[5] (that is, if P_1 plays perfectly against the computed P_2 strategy, then P_1 earns -37; if P_2 plays perfectly against the computed P_1 strategy, then P_2 earns 37). An abstraction of the game which is only 0.02% of the size of the full game produces a P_1 strategy that can be beaten by 112 mbb/h, and a P_2 strategy that can be beaten by 21 mbb/h. Still, the abstract strategy estimates that at equilibrium, P_2 can expect to win 35 mbb/h. So even though the abstraction produces a very poor estimate of the *strategy* σ^*, it produces a good estimate of the *value* of σ^*. In our experiments, we estimate $CBV^{\sigma_2^*}(I_1)$ by calculating a P_1 counterfactual best response *within the abstract game* to P_2's blueprint. We refer to this strategy as $\tilde{CBR}(\sigma_2)$ and its value in an infoset I_1 as $\tilde{CBV}^{\sigma_2}(I_1)$.

[5]In poker, the performance of one strategy against another depends on how much money is being wagered. For this reason, expected value and exploitability are measured in milli big blinds per hand (mbb/h). A big blind is the amount of money one of the players is required to put into the pot at the beginning of each hand.

We then use $\tilde{CBV}^{\sigma_2}(I_1)$ as the alternative payoff of I_1 in an augmented subgame. In other words, rather than calculate a P_1 counterfactual best response in the full game to P_2's blueprint strategy (which would be $CBR(\sigma_2)$), we instead calculate P_1's counterfactual best response where P_1 is constrained by the abstraction.

If the blueprint was produced by conducting T iterations of CFR in an abstract game, then one could instead simply use the final iteration's strategy σ_1^T, as this converges to a counterfactual best response within the abstract game. This is what we use in our experiments in this paper.

Using estimates of the values of σ^* tends to be do better than the theoretically safe options described in Section 4.[6]

C.1 Distributional Alternative Payoffs

One problem with existing safe subgame-solving techniques is that they may "overfit" to the alternative payoffs, even when we use estimates. Consider for instance a subgame with two different P_1 infosets I_1 and I'_1 at the top. Assume P_1's value for I_1 is estimated to be 1, and for I'_1 is 10. Now suppose during subgame solving, P_2 has a choice between two different strategies. The first sets P_1's value in the subgame for I_1 to 0.99 and for I'_1 to 9.99. The second slightly increases P_1's value for the subgame for I_1 to 1.01 but dramatically lowers the value for I'_1 to 0. The safe subgame-solving methods described so far would choose the first strategy, because the second strategy leaves one of the margins negative. However, intuitively, the second strategy is likely the better option, because it is more robust to errors in the model. For example, perhaps we are not confident that 10 is the exact value, but instead believe its true value is normally distributed with 10 as the mean and a standard deviation of 1. In this case, we would prefer the strategy that lowers the value for I'_1 to 0.

To address this problem, we introduce a way to incorporate the modeling uncertainty into the game itself. Specifically, we introduce a new augmented subgame that makes subgame solving more robust to errors in the model. This augmented subgame changes the augmented subgame used in subgame Resolving (shown in Figure 3b) so that the alternative payoffs are random variables, and P_1 is informed at the start of the augmented subgame of the values drawn from the random variables (but P_2 is not). The augmented subgame is otherwise identical. A visualization of this change is shown in Figure 6. As the distributions of the random variables narrow, the augmented subgame converges to the Resolve augmented subgame (but still maximizes the minimum margin when all margins are positive). As the distributions widen, P_2 seeks to maximize the sum over all margins, regardless of which are positive or negative.

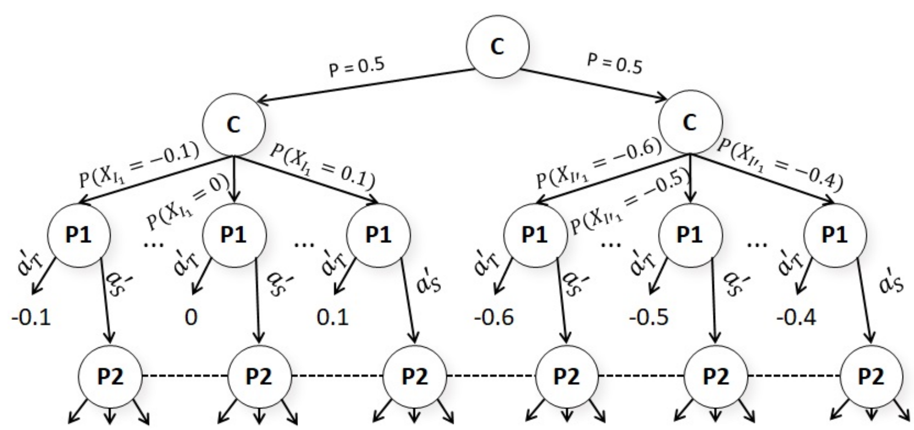


Figure 6: A visualization of the change in the augmented subgame from Figure 3b when using distributional alternative payoffs.

This modification makes the augmented subgame infinite in size because the random variables may be real-valued and P_1 could have a unique strategy for each outcome of the random variable.

[6]It is also possible to combine the safety of past approaches with some of the better performance of using estimates by adding the original Resolve conditions as additional constraints.

Fortunately, the special structure of the game allows us to arrive at a P_2 Nash equilibrium strategy for this infinite-sized augmented subgame by solving a much simpler gadget game.

The gadget game is identical to the augmented subgame used in Resolve subgame solving (shown in Figure 3b), except at each initial P_1 infoset $I_{r,1} \in S_r$, P_1 chooses action a'_S (that is, chooses to enter the subgame rather than take the alternative payoff) with probability $P\big(X_{I_1} \leq v(I_{r,1}, a'_S)\big)$, where $v(I_{r,1}, a'_S)$ is the expected value of action a'_S. (When solving via CFR, it is the expected value on each iteration, as described in CFR-BR [16]). This leads to Theorem 2, which proves that solving this simplified gadget game produces a P_2 strategy that is a Nash equilibrium in the infinite-sized augmented subgame illustrated in Figure 6.

Theorem 2. *Let S' be a Resolve augmented subgame and S'_r its root. Let S be a Distributional augmented subgame similar to S', except at each infoset $I_{r,1} \in S_r$, P_1 observes the outcome of a random variable X_{I_1} and the alternative payoff is equal to that outcome. If CFR is used to solve S' except that the action leading to S' is taken from each $I_{r,1} \in S'_r$ with probability $P\big(X_{I_1} \leq v^t(I_{r,1}, a'_S)\big)$, where $v^t(I_{r,1}, a'_S)$ is the value on iteration t of action a'_S, then the resulting P_2 strategy $\sigma_2^{S'}$ in S' is a P_2 Nash equilibrium strategy in S.*

Another option which also solves the game but has better empirical performance relies on the *softmax* (also known as *Hedge*) algorithm [18]. This gadget game is more complicated, and is described in detail in Appendix D. We use the softmax gadget game in our experiments.

The correct distribution to use for the random variables ultimately depends on the actual unknown errors in the model. In our experiments for this technique, we set $X_{I_1} \sim \mathcal{N}\big(\mu_{I_1}, s_{I_1}^2\big)$, where μ_I is the blueprint value (plus any gifts). s_{I_1} is set as the difference between the blueprint value of I_1, and the true (that is, unabstracted) counterfactual best response value of I_1. Our experiments show that this heuristic works well, and future research could yield even better options.

D Hedge for Distributional Subgame Solving

In this paper we use CFR [31] with Hedge in S_r, which allows us to leverage a useful property of the Hedge algorithm [18] to update all the infosets resulting from outcomes of X_{I_1} simultaneously.[7] When using Hedge, action a'_S in infoset $I_{r,1}$ in the augmented subgame is chosen on iteration t with probability $\frac{e^{\eta_t \hat{v}(I_{r,1}, a'_S)}}{e^{\eta_t \hat{v}(I_{r,1}, a'_S)} + e^{\eta_t \hat{v}(I_{r,1}, a'_T)}}$. Where $\hat{v}(I_{r,1}, a'_T)$ is the observed expected value of action a'_T, $\hat{v}(I_{r,1}, a'_S)$ is the observed expected value of action a'_S, and η_t is a tuning parameter. Since, action a'_S leads to identical play by both players for all outcomes of X, $\hat{v}(I_{r,1}, a'_S)$ is identical for all outcomes of X. Moreover, $\hat{v}(I_{r,1}, a'_T)$ is simply the outcome of X_{I_1}. So the probability that a'_S is taken across all infosets on iteration t is

$$\int_{-\infty}^{\infty} \frac{e^{\eta_t \hat{v}(I_{r,1}, a'_S)}}{e^{\eta_t \hat{v}(I_{r,1}, a'_S)} + e^{\eta_t x}} f_{X_{I_1}}(x) dx \tag{3}$$

where $f_{X_{I_1}}(x)$ is the pdf of X_{I_1}. In other words, if CFR is used to solve the augmented subgame, then the game being solved is identical to Figure 3b except that action a'_S is always chosen in infoset I_1 on iteration t with probability given by (3). In our experiments, we set the Hedge tuning parameter η as suggested in [3]: $\eta_t = \frac{\sqrt{\ln(|A(I_1)|)}}{3\sqrt{VAR(I_1)_t}\sqrt{t}}$, where $VAR(I_1)_t$ is the observed variance in the payoffs the infoset has received across all iterations up to t. In the subgame that follows S_r, we use CFR+ as the solving algorithm.

E Full Experimental Results

In tables 3, 4, and 5 we show the full results of our subgame solving experiments on various numbers of buckets.

[7] Another option is to apply CFR-BR [16] only at the initial P_1 nodes when deciding between a'_T and a'_S.

Small Flop Hold'em Flop Buckets:	200	2,000	30,000
Blueprint Strategy	886.9	373.7	91.28
Unsafe	146.8	39.58	5.514
Resolve	601.6	177.9	54.07
Maxmargin	300.5	139.9	43.43
Reach-Maxmargin	298.8	139.0	41.47
Reach-Maxmargin (not split)	248.7	98.07	25.88
Estimated	116.6	62.61	24.23
Estimated + Distributional	104.4	62.45	34.30
Reach-Estimated + Distributional	102.1	57.98	22.58
Reach-Estimated + Distributional (not split)	95.60	49.24	17.33

Table 3: Exploitability (evaluated in the game with no information abstraction) of subgame-solving in small Flop Texas hold'em.

Large Flop Hold'em Flop Buckets:	200	2,000	30,000
Blueprint Strategy	283.7	165.2	41.41
Unsafe	65.59	38.22	396.8
Resolve	179.6	101.7	23.11
Maxmargin	134.7	77.89	19.50
Reach-Maxmargin	134.0	72.22	18.80
Reach-Maxmargin (not split)	130.3	66.79	16.41
Estimated	52.62	41.93	30.09
Estimated + Distributional	49.56	38.98	10.54
Reach-Estimated + Distributional	49.33	38.52	9.840
Reach-Estimated + Distributional (not split)	49.13	37.22	8.777

Table 4: Exploitability (evaluated in the game with no information abstraction) of subgame-solving in large Flop Texas hold'em.

Turn Hold'em Turn Buckets:	200	2,000	20,000
Blueprint Strategy	684.6	465.1	345.5
Unsafe	130.4	85.95	79.34
Resolve	454.9	321.5	251.8
Maxmargin	427.6	299.6	234.4
Reach-Maxmargin	424.4	298.3	233.5
Reach-Maxmargin (not split)	333.4	229.4	175.5
Estimated	120.6	89.43	76.44
Estimated + Distributional	119.4	87.83	74.35
Reach-Estimated + Distributional	116.8	85.80	72.59
Reach-Estimated + Distributional (not split)	113.3	83.24	70.68

Table 5: Exploitability (evaluated in the game with no information abstraction) of subgame-solving in Turn Texas hold'em.

F Scaling of Gifts

To retain the theoretical guarantees of Reach subgame solving, one must ensure that the gifts assigned to reachable subgames do not (in aggregate) exceed the original gift. That is, if $g(I_1)$ is a gift at infoset I_1, we must ensure that $CBV^{\sigma_2^*}(I_1) \leq CBV^{\sigma_2}(I_1) + g(I_1)$. In this paper we accomplish this by increasing the margin of an infoset I_1', where $I_1 \sqsubseteq I_1'$, by at most $g(I_1)$. However, empirical performance may improve if the increase to margins due to gifts is scaled up by some factor. In most games we experimented on, exploitability decreased the further the gifts were scaled. However, Figure 7 shows one case in which we observe the exploitability increasing when the gifts are scaled up too far. The graph shows exploitability when the gifts are scaled by various factors. At 0, the algorithm is identical to Maxmargin. at 1, the algorithm is the theoretically correct form of Reach-Maxmargin. Optimal performance in this game occurs when the gifts are scaled by a factor of about $1,000$. Scaling the gifts by $100,000$ leads to performance that is worse than Maxmargin subgame solving. This empirically demonstrates that while scaling up gifts may lead to better performance in some cases (because an entire gift is unlikely to be used in every subgame that receives one), it may also lead to far worse performance in some cases.

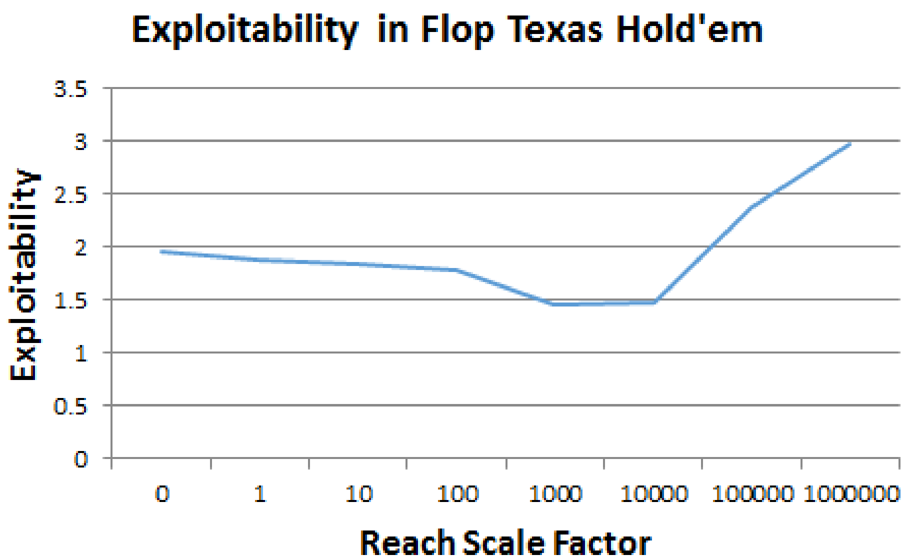


Figure 7: Exploitability in Flop Texas Hold'em of Reach-Maxmargin as we scale up the size of gifts.

G Rules for Poker Variants

Our experiments are conducted on heads-up no-limit Texas hold'em (HUNL), as well as smaller-scale variants we call no-limit flop hold'em (NLFH) and no-limit turn hold'em (NLTH). We begin by describing the rules of HUNL.

In the form of HUNL discussed in this paper, each player starts a hand with $20,000. One player is designated P_1, while the other is P_2. This assignment alternates between hands. HUNL consists of four rounds of betting. On a round of betting, each player can choose to either fold, call, or raise. If a player folds, that player immediately surrenders the pot to the opponent and the game ends. If a player calls, that players places a number of chips in the pot equal to the opponent's contribution. If a player raises, that player adds more chips to the pot than the opponent's contribution. A round of betting ends after a player calls. Players can continue to go back and forth with raises in a round until one of them runs out of chips.

If either player chooses to raise first in a round, they must raise a minimum of $100. If a player raises after another player has raised, that raise must be greater than or equal to the last raise. The maximum amount for a bet or raise is the remainder of that player's chip stack, which in our model is $20,000 at the beginning of a game.

At the start of HUNL, both players receive two private cards from a standard 52-card deck. P_1 must place a *big blind* of $100 in the pot, while P_2 must place a *small blind* of $50 in the pot. There is then a round of betting (the *preflop*), starting with P_2. When the round ends, three *community* cards are dealt face up between the players. There is then another round of betting (the *flop*), starting with P_1 this time. After the round of betting completes, another community card is dealt face up, and another round of betting commences starting with P_1 (the *turn*). Finally, one more community card is dealt face up, and a final betting round occurs (the *river*), again starting with P_1. If neither player folds before the final betting round completes, the player with the best five-card poker hand, constructed from their two private cards and the five face-up community cards, wins the pot. In the case of a tie, the pot is split evenly.

NLTH is similar to no-limit Texas hold'em except there are only three rounds of betting (the preflop, flop, and turn) in which there are two options for bet sizes. There are also only four community cards. NLFH is similar except there are only two rounds of betting (the preflop and flop), and three community cards.

We experiment with two versions of NLFH, one small and one large, which include only a few of the available actions in each infoset. The small game requires 1.1 GB to store the unabstracted strategy as double-precision floats. The large game requires 4 GB. NLTH requires 35 GB to store the unabstracted strategy.

H Proof of Theorem 1

Proof. Assume $M_r^{\sigma^S}(I_1) \geq 0$ for every infoset I_1 and assume $\pi_1^{BR(\sigma_2')}(I_1^*) > 0$ for some $I_1^* \in S_{top}$ and let $\epsilon = M_r(I_1^*)$. Define $\pi_{-1}^{\sigma}(I_1) = \sum_{h \in I_1} \pi_{-1}^{\sigma}(h)$ and define $\pi_{-1}^{\sigma}(I_1, I_1') = \sum_{h \in I_1, h' \in I_1'} \pi_{-1}^{\sigma}(h, h')$.

We show that for every P_1 infoset $I_1 \sqsubseteq I_1^*$ where $P(I_1) = P_1$,

$$CBV^{\sigma_2'}(I_1) \leq CBV^{\sigma_2^{-S}}(I_1) + \sum_{I_1'' \cdot a'' \sqsubseteq I_1 | P(I_1'') = P_1} \Big(\lfloor CBV^{\sigma_2^{-S}}(I_1'') - CBV^{\sigma_2^{-S}}(I_1'', a'') \rfloor \Big) - \sum_{h \in I_1, h^* \in I_1^*} \pi_{-1}^{\sigma_2}(h, h^*)\epsilon \quad (4)$$

By the definition of $M_r^{\sigma^S}(I_1^*)$ this holds for I_1^* itself. Moreover, the condition holds for every other $I_1 \in S_{top}$, because by assumption every margin is nonnegative and $\pi_{-1}^{\sigma_2}(I_1, I_1^*) = 0$ for any $I_1 \in S_{top}$ where $I_1 \neq I_1^*$. The condition also clearly holds for any I_1 with no descendants in S because then $\pi_{-1}^{\sigma_2}(I_1, I_1^*) = 0$ and $\sigma_2'(h) = \sigma_2^{-S}(h)$ in all P_2 nodes following I_1. This satisfies the base step. We now move on to the inductive step.

Let $Succ(I_1, a)$ be the set of earliest-reachable P_1 infosets following I_1 such that $P(I_1') = P_1$ for $I' \in Succ(I_1, a)$. Formally, $I_1' \in Succ(I_1, a)$ if $P(I_1') = P_1$ and $I_1 \cdot a \sqsubseteq I_1'$ and for any other $I_1'' \in Succ(I_1, a)$, $I_1'' \not\sqsubseteq I_1'$. Then

$$CBV^{\sigma_2'}(I_1, a) = CBV^{\sigma_2^{-S}}(I_1, a) + \sum_{I_1' \in Succ(I_1, a)} \pi_{-1}^{\sigma_2'}(I_1, I_1') (CBV^{\sigma_2'}(I_1') - CBV^{\sigma_2^{-S}}(I_1')) \quad (5)$$

Assume that every $I_1' \in Succ(I_1, a)$ satisfies (4). Then

$$CBV^{\sigma_2'}(I_1, a) \leq CBV^{\sigma_2^{-S}}(I_1, a) - \pi_{-1}^{\sigma_2}(I_1, I_1^*)\epsilon + \sum_{I_1' \in Succ(I_1, a)} \pi_{-1}^{\sigma_2}(I_1, I_1') \Big(\sum_{I_1'' \cdot a'' \sqsubseteq I_1' | P(I_1'') = P_1} \Big(\lfloor CBV^{\sigma_2^{-S}}(I_1'') - CBV^{\sigma_2^{-S}}(I_1'', a'') \rfloor \Big) \Big)$$

$$CBV^{\sigma_2'}(I_1, a) \leq CBV^{\sigma_2^{-S}}(I_1) - \Big(CBV^{\sigma_2^{-S}}(I_1) - CBV^{\sigma_2^{-S}}(I_1, a) \Big) - \pi_{-1}^{\sigma_2}(I_1, I_1^*)\epsilon + \sum_{I_1' \in Succ(I_1, a)} \pi_{-1}^{\sigma_2}(I_1, I_1') \Big(\sum_{I_1'' \cdot a'' \sqsubseteq I_1' | P(I_1'') = P_1} \Big(\lfloor CBV^{\sigma_2^{-S}}(I_1'') - CBV^{\sigma_2^{-S}}(I_1'', a'') \rfloor \Big) \Big)$$

Since $\lfloor CBV^{\sigma_2^{-S}}(I_1) - CBV^{\sigma_2^{-S}}(I_1, a) \rfloor \leq CBV^{\sigma_2^{-S}}(I_1) - CBV^{\sigma_2^{-S}}(I_1, a_1)$ so we get

$$CBV^{\sigma_2'}(I_1, a) \leq CBV^{\sigma_2^{-S}}(I_1) - \lfloor (CBV^{\sigma_2^{-S}}(I_1) - CBV^{\sigma_2^{-S}}(I_1, a) \rfloor - \pi_{-1}^{\sigma_2}(I_1, I_1^*)\epsilon + \sum_{I_1' \in Succ(I_1, a)} \pi_{-1}^{\sigma_2}(I_1, I_1') \Big(\sum_{I_1'' \cdot a'' \sqsubseteq I_1' | P(I_1'') = P_1} \Big(\lfloor CBV^{\sigma_2^{-S}}(I_1'') - CBV^{\sigma_2^{-S}}(I_1'', a'') \rfloor \Big) \Big)$$

$$CBV^{\sigma_2'}(I_1, a) \leq CBV^{\sigma_2^{-S}}(I_1) - \pi_{-1}^{\sigma_2}(I_1, I_1^*)\epsilon + \sum_{I_1' \in Succ(I_1, a)} \pi_{-1}^{\sigma_2}(I_1, I_1') \Big(\sum_{I_1'' \cdot a'' \sqsubseteq I_1 | P(I_1'') = P_1} \Big(\lfloor CBV^{\sigma_2^{-S}}(I_1'') - CBV^{\sigma_2^{-S}}(I_1'', a'') \rfloor \Big) \Big)$$

$$CBV^{\sigma'_2}(I_1, a_1) \leq CBV^{\sigma_2^{-S}}(I_1) - \pi_{-1}^{\sigma_2}(I_1, I_1^*)\epsilon + \sum_{I''_1 \cdot a'' \sqsubseteq I_1 | P(I''_1) = P_1} \Big(\lfloor CBV^{\sigma_2^{-S}}(I''_1) - CBV^{\sigma_2^{-S}}(I''_1, a''_1) \rfloor \Big)$$

Since $\pi_1^{BR(\sigma'_2)}(I_1^*) > 0$, and action a leads to I_1^*, so by definition of a best response, $CBV^{\sigma'_2}(I_1, a) = CBV^{\sigma'_2}(I_1)$. Thus,

$$CBV^{\sigma'_2}(I_1) \leq CBV^{\sigma_2^{-S}}(I_1) - \pi_{-1}^{\sigma_2}(I_1, I_1^*)\epsilon + \sum_{I''_1 \cdot a'' \sqsubseteq I_1 | P(I''_1) = P_1} \Big(\lfloor CBV^{\sigma_2^{-S}}(I''_1) - CBV^{\sigma_2^{-S}}(I''_1, a'') \rfloor \Big)$$

which satisfies the inductive step.

Applying this reasoning to the root of the entire game, we arrive at $exp(\sigma'_2) \leq exp(\sigma_2^{-S}) - \pi_{-1}^{\sigma_2}(I_1^*)\epsilon$. ☐

I Proof of Theorem 2

Proof. We prove inductively that using CFR in S' while choosing the action leading to S' from each $I_1 \in S'_r$ with probability $P\big(X_{I_1} \leq v^t(I_1, a'_S)\big)$ results in play that is identical to CFR in S and CFR-BR [16] in S_r, which converges to a Nash equilibrium.

For each P_2 infoset I'_2 in S' where $P(I'_2) = P_2$, there is exactly one corresponding infoset I_2 in S that is reached via the same actions, ignoring random variables. Each P_1 infoset I'_1 in S' where $P(I'_1) = P_1$ corresponds to a set of infosets in S that are reached via the same actions, where the elements in the set differ only by the outcome of the random variables. We prove that on each iteration, the instantaneous regret for these corresponding infosets is identical (and therefore the average strategy played in the P_2 infosets over all iterations is identical).

At the start of the first iteration of CFR, all regrets are zero. Therefore, the base case is trivially true. Now assume that on iteration t, regrets are identical for all corresponding infosets. Then the strategies played on iteration t in S are identical as well.

First, consider an infoset I'_1 in S' and a corresponding infoset I_1 in S. Since the remaining structure of the game is identical beyond I'_1 and I_1, and because P_2's strategies are identical in all P_2 infosets encountered, so the immediate regret for I'_1 and I_1 is identical as well.

Next, consider a P_1 infoset $I_{1,x}$ in S_r in which the random variable X_{I_1} has an observed value of x. Let the corresponding P_1 infoset in S'_r be I'_1. Since CFR-BR is played in this infoset, and since action a'_T leads to a payoff of x, so P_1 will choose action a'_S with probability 1 if $x \geq a'_T$ and with probability 0 otherwise. Thus, for all infosets in S_r corresponding to I'_1, action a'_S is chosen with probability $P\big(X_{I_1} \leq v(I_1, a'_S)\big)$.

Finally, consider a P_2 infoset I_2 in S and its corresponding infoset I'_2 in S'. Since in both cases action a'_T is taken in S_r with probability $P\big(X_{I_1} \leq v(I_1, a'_S)\big)$, and because P_1 plays identically between corresponding infosets in S and S', and because the structure of the game is otherwise identical, so the immediate regret for I'_1 and I_1 is identical as well. ☐