

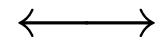
Focusing and higher-order abstract syntax

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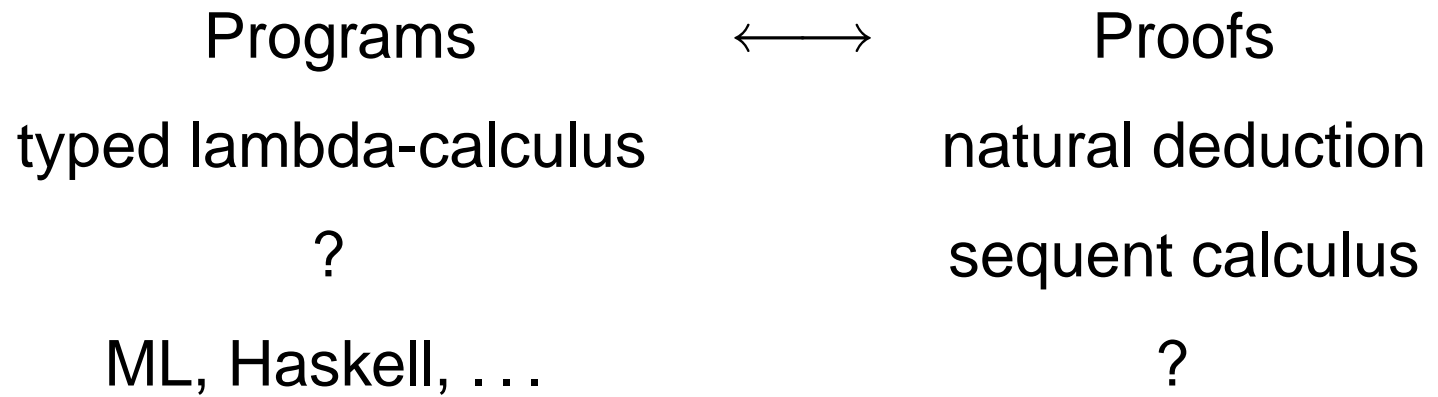
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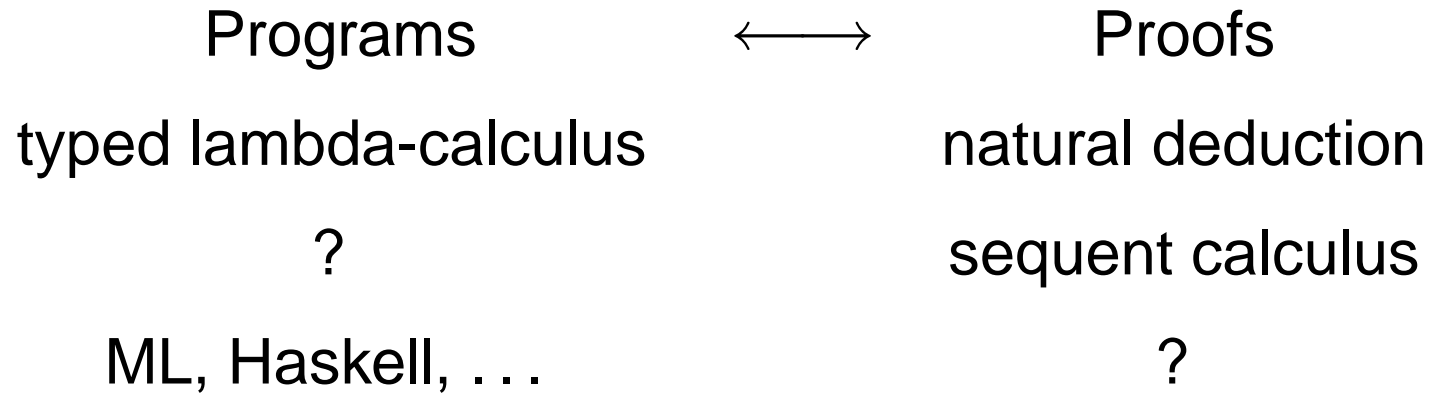
Programs



Proofs

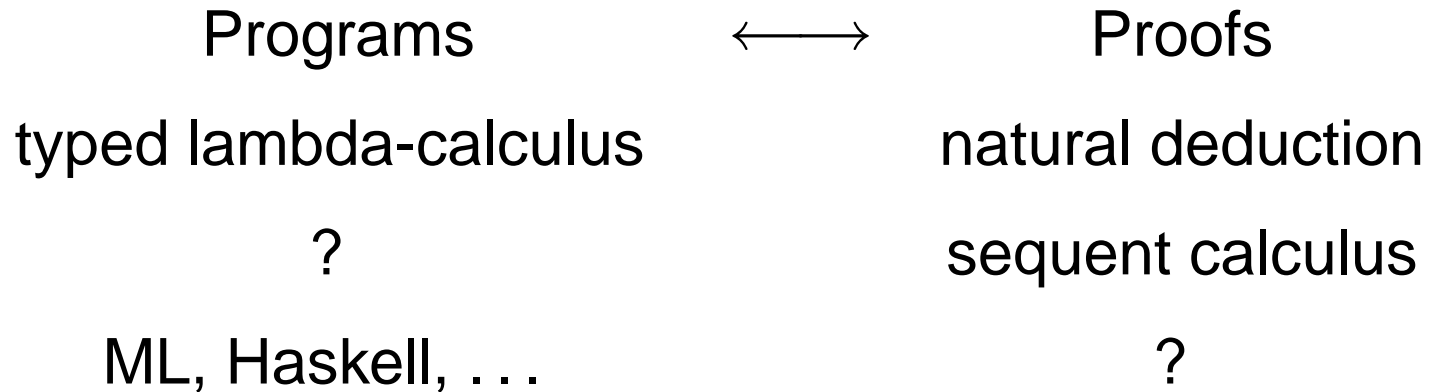
Programs \longleftrightarrow Proofs
typed lambda-calculus natural deduction





Practical features

1. explicit evaluation order
2. pattern-matching



Practical features

1. explicit evaluation order
2. pattern-matching

Claim 1: **focused** sequent calculus provides both 1 and 2

Claim 2: these features actually *simplify* logic & language

The bureaucracy of syntax

Two equivalent λ -calculus terms:

$\text{case}(\pi_1 x, y_1.\text{case}(\pi_2 x, z_1.e_1, z_2.e_2), y_2.\text{case}(\pi_2 x, z_1.e_3, z_2.e_4))$
 $\text{case}(\pi_2 x, z_1.\text{case}(\pi_1 x, y_1.e_1, y_2.e_3), z_2.\text{case}(\pi_1 x, y_1.e_2, y_2.e_4))$

Same term in ML:

```
case x of
  (Inl y1, Inl z1) => e1
| (Inl y1, Inr z2) => e2
| (Inr y2, Inl z1) => e3
| (Inr y2, Inr z2) => e4
```


Bottom-up proof-search

Naive bottom-up search:

1. Find a rule with goal as conclusion
2. Apply rule (premises become the new goals)
3. Upon failure, backtrack and try applying a different rule

Fact: naive bottom-up search is wildly inefficient for LL

Andreoli's solution: tame proof-search via **inversion** and **focus**

Inversion

A rule is *invertible* if its conclusion implies its premises

Every connective of LL is invertible on either left or right, e.g.:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} (\&R) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} (\otimes L)$$

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Inversion stage: chain application of invertible rules

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Inversion stage: chain application of invertible rules

$$\Gamma; (A \oplus B) \otimes (C \oplus D) \vdash E$$

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$$\frac{\frac{\Gamma; A, C \oplus D \vdash E \qquad \Gamma; B, C \oplus D \vdash E}{\Gamma; A \oplus B, C \oplus D \vdash E}}{\Gamma; (A \oplus B) \otimes (C \oplus D) \vdash E}$$

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$$\frac{\frac{\frac{\Gamma; A, C \vdash E \quad \Gamma; A, D \vdash E}{\Gamma; A, C \oplus D \vdash E} \quad \frac{\Gamma; B, C \vdash E \quad \Gamma; B, C \vdash E}{\Gamma; B, C \oplus D \vdash E}}{\Gamma; A \oplus B, C \oplus D \vdash E}}{\Gamma; (A \oplus B) \otimes (C \oplus D) \vdash E}$$

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Inversion stage: chain application of invertible rules

... and treat as one big step:

$$\frac{\Gamma, A, C \vdash E \quad \Gamma, A, D \vdash E \quad \Gamma, B, C \vdash E \quad \Gamma, B, D \vdash E}{\Gamma; (A \oplus B) \otimes (C \oplus D) \vdash E}$$

Flashback

Two equivalent λ -calculus terms:

$$\text{case}(\pi_1 x, y_1.\text{case}(\pi_2 x, z_1.e_1, z_2.e_2), y_2.\text{case}(\pi_2 x, z_1.e_3, z_2.e_4))$$
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```

Focus

Focus stage: chain application of non-invertible rules

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$$\Gamma \vdash [(A \oplus B) \otimes (C \oplus D)]$$

Focus

Focus stage: chain application of non-invertible rules

$$\frac{\Gamma_1 \vdash [A \oplus B] \quad \Gamma_2 \vdash [C \oplus D]}{\Gamma_1, \Gamma_2 \vdash [(A \oplus B) \otimes (C \oplus D)]}$$

Focus

Focus stage: chain application of non-invertible rules

$$\frac{\frac{\Gamma_1 \vdash [B]}{\Gamma_1 \vdash [A \oplus B]} \quad \Gamma_2 \vdash [C \oplus D]}{\Gamma_1, \Gamma_2 \vdash [(A \oplus B) \otimes (C \oplus D)]}$$

Focus

Focus stage: chain application of non-invertible rules

$$\frac{\frac{\Gamma_1 \vdash [B]}{\Gamma_1 \vdash [A \oplus B]} \quad \frac{\Gamma_2 \vdash [C]}{\Gamma_2 \vdash [C \oplus D]}}{\Gamma_1, \Gamma_2 \vdash [(A \oplus B) \otimes (C \oplus D)]}$$

Focus

Focus stage: chain application of non-invertible rules
... and treat as one big (nondeterministic) step:

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Theorem (Andreoli): focusing proof-search is complete.

Duality²

1. Inversion and focus are dual stages
2. They divide the connectives into two groups

Together these dualities form a *square of opposition*:

<i>polarity</i>		<i>inversion</i>	<i>focus</i>
<i>positive</i>	$\otimes, \oplus, 1, 0$	left	right
<i>negative</i>	$\&, \wp, \top, \perp$	right	left

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Girard '93: dualities extend beyond LL by *polarization*...

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Girard '93: dualities extend beyond LL by *polarization* . . .
... i.e., by fixing evaluation order

What you'll find in the paper

A new formulation of focusing for intuitionistic logic with. . .

- big-step **inversion** and **focus** through higher-order rules
- A “connective-blind” proof of cut-elimination!

A Curry-Howard interpretation incorporating. . .

- CBV functions, strict products and sums, recursive types
- Pattern-matching “for free” through a HO encoding:

fnc = **pat** \rightarrow **exp** **val** = **pat** \times **sub**

- A connective-blind proof of type safety

A formalization in Coq

A very quick sketch

Connectives of the day

Polarized intuitionistic connectives

Positive: $\otimes, \oplus, 1, 0$ (strict products and sums)

Negative: $\overset{v}{\rightarrow}$ (CBV function space)

(Same entailments as IL, different proofs.)

Notation for polarized formulas:

$P, Q ::= X \mid P \otimes Q \mid P \oplus Q \mid 0 \mid 1 \mid N$ ← implicit coercion

$N ::= P \overset{v}{\rightarrow} Q$

In paper: $P ::= \dots \mid \mu X.P$

“Hidden” slides (not in paper): $N, M ::= \dots \mid N \overset{n}{\rightarrow} M \mid N \& M \mid \dots$

Linear right rules \approx Pattern typing

Linear contexts $\Delta ::= \cdot \mid X, \Delta \mid N, \Delta$

$$\boxed{\Delta \Rightarrow P}$$

$$\overline{X \Rightarrow X} \quad \overline{N \Rightarrow N}$$

$$\overline{\cdot \Rightarrow 1} \quad \frac{\Delta_1 \Rightarrow P \quad \Delta_2 \Rightarrow Q}{\Delta_1, \Delta_2 \Rightarrow P \otimes Q}$$

(no rule for 0)

$$\frac{\Delta \Rightarrow P}{\Delta \Rightarrow P \oplus Q} \quad \frac{\Delta \Rightarrow Q}{\Delta \Rightarrow P \oplus Q}$$

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Linear right rules \approx Pattern typing

Linear contexts $\Delta ::= \cdot \mid x : X, \Delta \mid f : N, \Delta$

$$\boxed{\Delta \Rightarrow p : P}$$

$$\frac{}{x : X \Rightarrow x : X} \quad \frac{}{f : N \Rightarrow f : N}$$

$$\frac{}{\cdot \Rightarrow () : 1} \quad \frac{\Delta_1 \Rightarrow p_1 : P \quad \Delta_2 \Rightarrow p_2 : Q}{\Delta_1, \Delta_2 \Rightarrow (p_1, p_2) : P \otimes Q}$$

(no rule for 0) $\frac{\Delta \Rightarrow p : P}{\Delta \Rightarrow \text{inl } p : P \oplus Q} \quad \frac{\Delta \Rightarrow p : Q}{\Delta \Rightarrow \text{inr } p : P \oplus Q}$

Usual pattern restrictions emerge!

Focused [sequent/lambda]-calculus

Unrestricted contexts $\Gamma ::= \cdot \mid \Gamma, \Delta$

Judgments:

$\Gamma \vdash [P]$ right-focus

$\Gamma; P \vdash Q$ left-inversion

$\Gamma \vdash P$ unfocused

$\Gamma \vdash \Delta$ multiple (conjoined) conclusions

Focused [sequent/lambda]-calculus

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$$\Gamma; P \vdash Q$$

$$\Gamma \vdash P$$

$$\Gamma \vdash \Delta$$

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$\Gamma \vdash V : [P]$ value

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$\Gamma \vdash P$

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$\Gamma \vdash V : [P]$ value

$\Gamma \vdash F : P > Q$ (CBV) function

$\Gamma \vdash P$

$\Gamma \vdash \Delta$

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$\Gamma \vdash V : [P]$	value
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$\Gamma \vdash E : P$	expression
$\Gamma \vdash \Delta$	

Focused [sequent/lambda]-calculus

Unrestricted contexts $\Gamma ::= \cdot \mid \Gamma, \Delta$

Judgments:

$\Gamma \vdash V : [P]$	value
$\Gamma \vdash F : P > Q$	(CBV) function
$\Gamma \vdash E : P$	expression
$\Gamma \vdash \sigma : \Delta$	substitution

Untyped syntax

Canonical fragment (β -reduced, η -long):

$$V ::= p[\sigma]$$

$$F ::= \lambda\phi$$

where $\phi ::= (p_1 \mapsto E_1 \mid \dots \mid p_n \mapsto E_n)$

$$E ::= V \mid F(g(V))$$

$$\sigma ::= \cdot \mid (y/x, \sigma) \mid (F/f, \sigma)$$

Identity and cut principles:

$$F ::= \dots \mid \text{id} \mid f$$

$$E ::= \dots \mid F(V) \mid F(E)$$

Right-focus \approx Value typing

$$\boxed{\Gamma \vdash [P]}$$

$$\frac{\Delta \Rightarrow P \quad \Gamma \vdash \Delta}{\Gamma \vdash [P]}$$

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$$\Gamma \vdash V : [P]$$

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$$\Gamma \vdash V : [P]$$

$$\frac{\Delta \Rightarrow p : P \quad \Gamma \vdash \sigma : \Delta}{\Gamma \vdash p[\sigma] : [P]}$$

Right-focus \approx Value typing

$$\Gamma \vdash V : [P]$$

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Yes, this is really a “value” in the ML sense...

(fn x => x*x, fn x => x-3)

Right-focus \approx Value typing

$$\Gamma \vdash V : [P]$$

$$\frac{\Delta \Rightarrow p : P \quad \Gamma \vdash \sigma : \Delta}{\Gamma \vdash p[\sigma] : [P]}$$

Yes, this is really a “value” in the ML sense...

$$(f, g)[(fn x => x*x)/f, (fn x => x-3)/g]$$

Left-inversion \approx CBV function typing

$$\boxed{\Gamma; P \vdash Q}$$

$$\frac{\forall(\Delta \Rightarrow P) : \Gamma, \Delta \vdash Q}{\Gamma; P \vdash Q}$$

Left-inversion \approx CBV function typing

$$\boxed{\Gamma \vdash F : P > Q}$$

$$\frac{\forall(\Delta \Rightarrow P) : \Gamma, \Delta \vdash Q}{\Gamma; P \vdash Q}$$

Left-inversion \approx CBV function typing

$$\boxed{\Gamma \vdash F : P > Q}$$

$$\frac{\forall(\Delta \Rightarrow p : P) : \Gamma, \Delta \vdash \phi(p) : Q}{\Gamma \vdash (\lambda\phi) : P > Q}$$

Left-inversion \approx CBV function typing

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ϕ a (partial) map from patterns to expressions!

In Coq, define `Lam : (pat \rightarrow exp) \rightarrow fnc`

Left-inversion \approx CBV function typing

$$\boxed{\Gamma \vdash F : P > Q}$$

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ϕ a (partial) map from patterns to expressions!

In Coq, define `Lam : (pat \rightarrow exp) \rightarrow fnc`

Hmm...?!

A more abstract syntax

The basic premise of this higher-order encoding is that the static and dynamic semantics of a language shouldn't care so much about the details of how a function is written down, but just get to the point about what expression is evaluated for each pattern. (Note this is *not* the same as only caring about the mapping from values to results!)

Several people have complained that calling this sort of encoding “higher-order abstract syntax” is misleading, since HOAS has a well-established usage referring to meta-level *substitution functions* used to encode object-level binders. They are probably right that my choice of terminology was confusing—sorry! Maybe it should be “abstract higher-order syntax”?...

Small-step semantics as cut-elimination

$$\text{id}(V) \rightsquigarrow V$$

$\frac{\phi(p) \text{ defined}}{(\lambda\phi)(p[\sigma]) \rightsquigarrow \phi(p)[\sigma]}$

$$\frac{E \rightsquigarrow E'}{F(E) \rightsquigarrow F(E')}$$

Theorem: Type Safety

Proof does not mention (positive) connectives!

Cf. Gentzen/Prawitz/Dummett proof-theoretic semantics

Coq encoding

“Curry-style” (untyped syntax, type system, opsem, type safety):

<http://www.cs.cmu.edu/~noam/research/focusing.tar>

“Church-style” (typed syntax only):

<http://www.cs.cmu.edu/~noam/research/focus-church.v>

“Church v2.0” (with uniform treatment of negative types):

<http://www.cs.cmu.edu/~noam/research/focusing2.v>

Define functions using Coq’s built-in pattern-matching!

(But use de Bruijn indices for binding, alas...)

Coq example: plus

```
Definition plus : fnc :=
  Fix (Lam (fun p => match p with
    | Pair m Ze => Return (Val m (Sub nil))
    | Pair m (Su n) =>
      EComp
        (Lam (fun n' =>
          Return (Val (Su n')
            (Sub nil))))
        (1,0) (Val (Pair m n) (Sub nil))
    | _ => EFail
  end)).
```

Conclusions

Focusing is awesome!

Focusing is awesome!

Why?

1. **Pattern-matching** \approx **focus** and **inversion**
2. **Explicit evaluation order** \approx **polarity**

Future Work

The computational world of polarized logic & type theory

- Mixed polarities = mixed evaluation strategies (Haskell?)
- Polymorphism & intersection/union types
- Dependent types

Higher-order encodings

- What is the relationship between AHOS and HOAS?
- Can they be combined in one logical framework?
- See Brigitte's talk! And paper with Licata & Harper...

Thanks!

Recursion and recursive types

Recursion:

$$\frac{\Gamma, f : P \xrightarrow{v} Q \vdash F : P > Q}{\Gamma \vdash \text{fix } f.F : P > Q} \quad (\text{fix } f.F)(V) \rightsquigarrow ([\text{fix } f.F / f]F)(V)$$

Recursive types:

$$\frac{\Delta \Rightarrow p : P[\mu X.P / X]}{\Delta \Rightarrow \text{fold}(p) : \mu X.P}$$

A uniform treatment of negative types

Make coercions explicit: $\downarrow N$ positive, $\uparrow P$ negative

$$P \xrightarrow{v} Q = P \rightarrow \uparrow Q \quad N \xrightarrow{n} M = \downarrow N \rightarrow M$$

Define negative connectives through linear left rules:

$$\boxed{\Delta; N \Rightarrow \gamma}$$

$$\frac{}{\cdot; \uparrow P \Rightarrow P} \quad \frac{\Delta_1 \Rightarrow P \quad \Delta_2; N \Rightarrow \gamma}{\Delta_1, \Delta_2; P \rightarrow N \Rightarrow \gamma}$$

$$\frac{\Delta; N \Rightarrow \gamma}{\Delta; N \& M \Rightarrow \gamma} \quad \frac{\Delta; M \Rightarrow \gamma}{\Delta; N \& M \Rightarrow \gamma}$$

Curry-Howard: destructor patterns

General intuitionistic focusing

Right-focus and left-inversion:

$$\frac{\Delta \Rightarrow P \quad \Gamma \vdash \Delta}{\Gamma \vdash [P]} \quad \frac{\forall(\Delta \Rightarrow P) : \Gamma, \Delta \vdash \gamma}{\Gamma; P \vdash \gamma}$$

Right-inversion and left-focus:

$$\frac{\forall(\Delta; N \Rightarrow \gamma) : \Gamma, \Delta \vdash \gamma}{\Gamma \vdash N} \quad \frac{\Delta; N \Rightarrow \gamma_0 \quad \Gamma \vdash \Delta \quad \Gamma; \gamma_0 \vdash \gamma}{\Gamma; [N] \vdash \gamma}$$

Unfocused sequents:

$$\frac{\Gamma \vdash [P]}{\Gamma \vdash P} \quad \frac{N \in \Gamma \quad \Gamma; [N] \vdash \gamma}{\Gamma \vdash \gamma}$$