Focusing with higher-order rules

Noam Zeilberger

Carnegie Mellon University

28 February 2008

A purely interactive approach to logic

« Interactive » could suggest yet-one-more-game-semantics : but the material presented here is neither syntax nor semantics, moreover the word *purely* suggests a distance with the mere idea of game : there is no rule —or no referee, if you prefer— like in real life. And *logic*, without « s », is for what should be the most natural thing in nature —something too often presented as the most artificial one.

The monograph ends with a dictionary, discussing these issues : sort of final introduction, since one can only introduce to known material. For instance if you go to DIALECTICS you will understand the word

Ludics

which is the real alternative title, the very name of the new area.

The novelty of ludics is conveyed by our title

Locus Solum

after the book by Raymond Roussel, *Locus Solus*, i.e., « solitary place ». *Locus Solum* means something like

Only the location matters

for the results presented here establish the pregnancy of location, the *locus*, in logic. As you will see, the irruption of the *locus* by no way weakens or dilutes logical principles : they just become different, more harmonious, and stronger. Moreover the logic-we-used-to-know-and-love is still present, but it now gets a specific name, *spiritual logic* : ludics created spiritual logic in the same way Brouwer created classical logic and Luther catholicism.

A purely interactive approach to logic

« Interactive » could suggest yet-one-more-game-semantics : but the material presented here is neither syntax nor semantics, moreover the word *purely* suggests a distance with the mere idea of game : there is no rule —or no referee, if you prefer— like in real life. And *logic*, without « s », is for what should be the most natural thing in nature —something too often presented as the most artificial one.

The monograph ends with a dictionary, discussing these issues : sort of final introduction, since one can only introduce to known material. For instance if you go to DIALECTICS you will understand the word

Ludics

which is the real alternative title, the very name of the new area.

The novelty of ludics is conveyed by our title

Locus Solum

after the book by Raymond Roussel, *Locus Solus*, i.e., « solitary place ». *Locus Solum* means something like

Only the location matters

for the results presented here establish the pregnancy of location, the *locus*, in logic. As you will see, the irruption of the *locus* by no way weakens or dilutes logical principles : they just become different, more harmonious, and stronger. Moreover the logic-we-used-to-know-and-love is still present, but it now gets a specific name, *spiritual logic* : ludics created spiritual logic in the same way Brouwer created classical logic and Luther catholicism.

Only the pattern matters

Being vs doing

What is polarity? One possible interpretation:

- Positive = "defined" by verification (intro rule)
- Negative = "defined" by use (elim rule)

This kind of duality is an old idea. . .

- Brouwer, Wittgenstein, Dummett, Martin-Löf, . . .
- Abramsky, Melliès, . . .
- Curien & Herbelin, Selinger, P-B Levy, McBride, . . .

A new idea. . .

Build focused sequent calculus in two stages

1. Define restricted, linear entailment (“pattern-typing”)
2. Define arbitrary entailment (“program-typing”)

Duality made explicit

- Positive = constructor patterns
- Negative = destructor patterns
- Focus = $\exists$ pattern, Inversion = $\forall$ patterns

Curry-Howard: pattern-matching and evaluation order

. . . but kind of an old idea

Buchholz' Ω-rule and IID

Schroeder-Heister's definitional reflection

More on these connections later. . .

Initial setting

Polarized intuitionistic logic:

$$\begin{aligned} A^+, B^+ \;::=\;& 1 \mid A^+ \otimes B^+ \mid 0 \mid A^+ \oplus B^+ \mid \mathbb{N} \mid \ldots \\ & \mid X^+ \mid {\downarrow} A^- \end{aligned}$$

$$\begin{aligned} A^-, B^- \;::=\;& \top \mid A^- \& B^- \mid A^+ \to B^- \mid {}^{\mathbb{N}}A^- \mid \ldots \\ & \mid X^- \mid {\uparrow} A^+ \end{aligned}$$

Intuitionistic restriction only for intuition. . .

Hypotheses, conclusions, contexts

Hypothesis	h	$::= X^+ \mid A^-$
Conclusion	c	$::= X^- \mid A^+$
Linear Context	Δ	$::= \cdot \mid \Delta, h$

```
Inductive hyp : Set :=
  PAtomH : atom -> hyp | NegH : neg -> hyp.
Inductive conc : Set :=
  NAtomC : atom -> conc | PosC : pos -> conc.
Definition linctx := list hyp.
```

Patterns

Constructor patterns

Positive connectives *defined* by judgment $\boxed{\Delta \Vdash A^+}$

$$\frac{}{X^+ \Vdash X^+} \qquad \frac{}{A^- \Vdash {\downarrow}A^-}$$

$$\frac{}{\cdot \Vdash 1} \qquad \frac{\Delta_1 \Vdash A^+ \quad \Delta_2 \Vdash B^+}{\Delta_1, \Delta_2 \Vdash A^+ \otimes B^+}$$

(no rule for 0) $$\frac{\Delta \Vdash A^+}{\Delta \Vdash A^+ \oplus B^+} \qquad \frac{\Delta \Vdash B^+}{\Delta \Vdash A^+ \oplus B^+}$$

. . . .

$$\frac{}{\cdot \Vdash \mathbb{N}} \qquad \frac{\Delta \Vdash \mathbb{N}}{\Delta \Vdash \mathbb{N}}$$

Coq...

```
Inductive patP : linctx -> pos -> Set :=
  | c_avar : forall X⁺,
      patP [ PAtomH X ] (PAtom X)
  | c_nvar : forall A⁻,
      patP [ NegH A⁻ ] (↓ A⁻)
  | c_unit : patP nil One
  | c_pair : forall Δ1 Δ2 A⁺ B⁺,
      patP Δ1 A⁺ -> patP Δ2 B⁺ ->
      patP (Δ1 ++ Δ2) (A⁺ ⊗ B⁺)
  | c_in1 : forall Δ A⁺ B⁺,
      patP Δ A⁺ -> patP Δ (A⁺ ⊕ B⁺)
  | c_in2 : forall Delta A⁺ B⁺,
      patP Δ B⁺ -> patP Δ (A⁺ ⊕ B⁺)
  | ...
```

Destructor patterns

Negative connectives *defined* by judgment $\boxed{\Delta; A^- \Vdash c}$

$$\frac{}{\cdot; X^- \Vdash X^-} \qquad \frac{}{\cdot; {\uparrow}A^+ \Vdash A^+}$$

(no rule for $\top$)

$$\frac{\Delta; A^- \Vdash c}{\Delta; A^- \& B^- \Vdash c} \qquad \frac{\Delta; B^- \Vdash c}{\Delta; A^- \& B^- \Vdash c}$$

$$\frac{\Delta_1 \Vdash A^+ \quad \Delta_2; B^- \Vdash c}{\Delta_1, \Delta_2; A^+ \to B^- \Vdash c}$$

. . . .

$$\frac{\Delta; A^- \Vdash c}{\Delta; {}^{\mathbb{N}}A^- \Vdash c} \qquad \frac{\Delta; {}^{\mathbb{N}}A^- \Vdash c}{\Delta; {}^{\mathbb{N}}A^- \Vdash c}$$

Coq…

```
Inductive patN : linctx -> neg -> conc -> Set
  | d_aid : forall X⁻,
      patN nil (NAtom X⁻) (NAtomC X⁻)
  | d_pid : forall A⁺,
      patN nil (↑ A⁺) (PosC A⁺)
  | d_pi1 : forall Δ A⁻ B⁻ c,
      patN Δ A⁻ c ->
      patN Δ (A⁻ & B⁻) c
  | d_pi2 : forall Δ A⁻ B⁻ c,
      patN Δ B⁻ c ->
      patN Δ (A⁻ & B⁻) c
  | d_app : forall Δ1 Δ2 A⁺ B⁻ c,
      patP Δ1 A⁺ -> patN Δ2 B⁻ c ->
      patN (Δ1 ++ Δ2) (A⁺ → B⁻) c
  | ...
```

</connectives>

Focusing

Contexts, judgments

Unrestricted contexts $\Gamma ::= \cdot \mid \Gamma, \Delta$

$\Gamma \vdash [A^+]$	right-focus
$\Gamma; \mathcal{c}_0 \vdash \mathcal{c}$	left-inversion
$\Gamma \vdash \mathcal{h}$	right-inversion
$\Gamma; [A^-] \vdash \mathcal{c}$	left-focus
$\Gamma \vdash \mathcal{c}$	unfocused
$\Gamma \vdash \Delta$	substitution

Right-focus (positive value)

$$\boxed{\Gamma \vdash [A^+]}$$

$$\frac{\Delta \Vdash A^+ \quad \Gamma \vdash \Delta}{\Gamma \vdash [A^+]}$$

Right-focus (positive value)

$$\boxed{\Gamma \vdash [A^+]}$$

$$\frac{\Delta \Vdash A^+ \quad \Gamma \vdash \Delta}{\Gamma \vdash [A^+]}$$

(C-H: value factors as pattern with substitution)

Right-focus (positive value)

$$\boxed{\Gamma \vdash [A^+]}$$

$$\frac{\Delta \Vdash A^+ \quad \Gamma \vdash \Delta}{\Gamma \vdash [A^+]}$$

(C-H: value factors as pattern with substitution)

$$\frac{\Gamma \vdash B_1^-, B_2^-}{\Gamma \vdash [\downarrow A^- \oplus (\downarrow B_1^- \otimes \downarrow B_2^-)]} (B_1^-, B_2^- \Vdash \text{-})$$

Right-focus (positive value)

$$\boxed{\Gamma \vdash [A^+]}$$

$$\frac{\Delta \Vdash A^+ \quad \Gamma \vdash \Delta}{\Gamma \vdash [A^+]}$$

(C-H: value factors as pattern with substitution)

$$\frac{\Gamma \vdash B_1^- \quad \Gamma \vdash B_2^-}{\Gamma \vdash [{\downarrow}A^- \oplus ({\downarrow}B_1^- \otimes {\downarrow}B_2^-)]} \; (B_1^-, B_2^- \Vdash \_)$$

Left-inversion (positive continuation)

$$\boxed{\Gamma; c_0 \vdash c}$$

$$\frac{}{\Gamma; X^- \vdash X^-} \qquad \frac{\forall(\Delta \Vdash A^+) : \quad \Gamma, \Delta \vdash c}{\Gamma; A^+ \vdash c}$$

Left-inversion (positive continuation)

$$\boxed{\Gamma; c_0 \vdash c}$$

$$\frac{}{\Gamma; X^- \vdash X^-} \qquad \frac{\forall(\Delta \Vdash A^+) : \quad \Gamma, \Delta \vdash c}{\Gamma; A^+ \vdash c}$$

(C-H: continuation defined by “abstract higher-order syntax”)

Left-inversion (positive continuation)

$$\boxed{\Gamma; c_0 \vdash c}$$

$$\frac{}{\Gamma; X^- \vdash X^-} \qquad \frac{\forall(\Delta \Vdash A^+) : \quad \Gamma, \Delta \vdash c}{\Gamma; A^+ \vdash c}$$

(C-H: continuation defined by “abstract higher-order syntax”)

$$\frac{\Gamma, A^- \vdash c \quad \Gamma, B_1^-, B_2^- \vdash c}{\Gamma; {\downarrow}A^- \oplus ({\downarrow}B_1^- \otimes {\downarrow}B_2^-) \vdash c}$$

Left-inversion (positive continuation)

$$\boxed{\Gamma; c_0 \vdash c}$$

$$\frac{}{\Gamma; X^- \vdash X^-} \qquad \frac{\forall(\Delta \Vdash A^+) : \quad \Gamma, \Delta \vdash c}{\Gamma; A^+ \vdash c}$$

This is Buchholz' Ω-rule!

(As special case, ω-rule for $\mathbb{N}$)

(C-H: continuation defined by “abstract higher-order syntax”)

$$\frac{\Gamma, A^- \vdash c \quad \Gamma, B_1^-, B_2^- \vdash c}{\Gamma; {\downarrow}A^- \oplus ({\downarrow}B_1^- \otimes {\downarrow}B_2^-) \vdash c}$$

Coq...

```
Inductive rfoc : intctx -> pos -> Set :=
  | ValP : forall Γ A⁺ Δ,
        patP Δ A⁺ -> satctx Γ Δ ->
        rfoc Γ A⁺

with linv : intctx -> conc -> conc -> Set :=
  | IdXN : forall Γ X,
        linv Γ (NAtomC X) (NAtomC X)
  | ConP : forall Γ A⁺ c,
        (forall Δ,
            patP Δ A⁺ -> unfoc (Δ :: Γ) c) ->
        linv Γ (PosC A⁺) c

with ...
```

Right-inversion (negative value)

$$\boxed{\Gamma \vdash h}$$

$$\frac{X^+ \in \Gamma}{\Gamma \vdash X^+} \qquad \frac{\forall(\Delta; A^- \Vdash c) : \quad \Gamma, \Delta \vdash c}{\Gamma \vdash A^-}$$

Right-inversion (negative value)

$$\boxed{\Gamma \vdash \boldsymbol{h}}$$

$$\frac{X^+ \in \Gamma}{\Gamma \vdash X^+} \qquad \frac{\forall(\Delta; A^- \Vdash \boldsymbol{c}) : \quad \Gamma, \Delta \vdash \boldsymbol{c}}{\Gamma \vdash A^-}$$

(C-H: lazy value, defined by matching against observations)

Right-inversion (negative value)

$$\boxed{\Gamma \vdash h}$$

$$\frac{X^+ \in \Gamma}{\Gamma \vdash X^+} \qquad \frac{\forall(\Delta; A^- \Vdash c) : \quad \Gamma, \Delta \vdash c}{\Gamma \vdash A^-}$$

(C-H: lazy value, defined by matching against observations)

$$\frac{\Gamma, A^- \vdash B_1^+ \quad \Gamma, A^- \vdash B_2^+}{\Gamma \vdash \downarrow A^- \to \uparrow B_1^+ \& \uparrow B_2^+}$$

Left-focus (negative continuation)

$$\boxed{\Gamma; [A^-] \vdash c}$$

$$\frac{\Delta; A^- \Vdash c_0 \quad \Gamma \vdash \Delta \quad \Gamma; c_0 \vdash c}{\Gamma; [A^-] \vdash c}$$

Left-focus (negative continuation)

$$\boxed{\Gamma; [A^-] \vdash c}$$

$$\frac{\Delta; A^- \Vdash c_0 \quad \Gamma \vdash \Delta \quad \Gamma; c_0 \vdash c}{\Gamma; [A^-] \vdash c}$$

(C-H: continuation for a lazy value)

Left-focus (negative continuation)

$$\boxed{\Gamma; [A^-] \vdash c}$$

$$\frac{\Delta; A^- \Vdash c_0 \quad \Gamma \vdash \Delta \quad \Gamma; c_0 \vdash c}{\Gamma; [A^-] \vdash c}$$

(C-H: continuation for a lazy value)

$$\frac{\Gamma \vdash A^- \quad \Gamma; B_1^+ \vdash c}{\Gamma; [\downarrow A^- \to \uparrow B_1^+ \& \uparrow B_2^+] \vdash c} \; (A^-;\_ \Vdash B_1^+)$$

Unfocused sequents and substitutions

$$\frac{\Gamma \vdash [A^+]}{\Gamma \vdash A^+} \qquad \frac{A^- \in \Gamma \quad \Gamma; [A^-] \vdash c}{\Gamma \vdash c}$$

. . . .

$$\frac{}{\Gamma \vdash \cdot} \qquad \frac{\Gamma \vdash \Delta \quad \Gamma \vdash h}{\Gamma \vdash \Delta, h}$$

(Asymmetry of intuitionistic logic)

Properties

Identity principles

1. $\Gamma; c \vdash c$
2. If $h \in \Gamma$ then $\Gamma \vdash h$
3. $\Gamma, \Delta \vdash \Delta$

Defined mutually, e.g. (2) reduces to (1) and (3):

$$\dfrac{\forall(\Delta; A^- \Vdash c) : \quad \dfrac{\dfrac{\Gamma, \Delta \vdash \Delta \quad \Gamma, \Delta; c \vdash c}{\Gamma, \Delta; [A^-] \vdash c}\,(\Delta; A^- \Vdash c)}{\Gamma, \Delta \vdash c}\,(A^- \in \Gamma)}{\Gamma \vdash A^-}$$

Cut principles

1. If $\Gamma \vdash [A^+]$ and $\Gamma; A^+ \vdash c$ then $\Gamma \vdash c$
2. If $\Gamma \vdash A^-$ and $\Gamma; [A^-] \vdash c$ then $\Gamma \vdash c$
3. (a) If $\Gamma \vdash c_0$ and $\Gamma; c_0 \vdash c$ then $\Gamma \vdash c$
   (b) If $\Gamma; [A^-] \vdash c_0$ and $\Gamma; c_0 \vdash c$ then $\Gamma; [A^-] \vdash c$
   (c) If $\Gamma; c_1 \vdash c_0$ and $\Gamma; c_0 \vdash c$ then $\Gamma; c_1 \vdash c$
4. If $\Gamma \vdash \Delta$ and $\Gamma, \Delta \vdash J$ then $\Gamma \vdash J$

For the proof-theorists:

(1) and (2) are *principal* cuts

(3) are *left-commutative* cuts

(4) are *right-commutative* cuts

Cut principles

$$\frac{\Delta \Vdash A^+ \quad \Gamma \vdash \Delta}{\Gamma \vdash [A^+]} \quad \text{cut} \quad \frac{\forall(\Delta \Vdash A^+) : \quad \Gamma, \Delta \vdash c}{\Gamma; A^- \vdash c}$$

$$\rightsquigarrow$$

Cut principles

$$\frac{\Delta \Vdash A^+ \quad \Gamma \vdash \Delta}{\Gamma \vdash [A^+]} \quad \text{cut} \quad \frac{\forall(\Delta \Vdash A^+) : \quad \Gamma, \Delta \vdash c}{\Gamma; A^- \vdash c}$$

$$\rightsquigarrow$$

$$\Gamma \vdash \Delta \quad \text{cut} \quad \Gamma, \Delta \vdash c$$

Cut principles

$$\dfrac{\Delta \Vdash A^+ \quad \Gamma \vdash \Delta}{\Gamma \vdash [A^+]} \quad \text{cut} \quad \dfrac{\forall(\Delta \Vdash A^+) : \quad \Gamma, \Delta \vdash c}{\Gamma; A^- \vdash c}$$

$$\rightsquigarrow$$

$$\Gamma \vdash \Delta \quad \text{cut} \quad \Gamma, \Delta \vdash c$$

. . . .

$$\dfrac{\Gamma \vdash \Delta \quad \Gamma \vdash A^-}{\Gamma \vdash \Delta, A^-} \quad \text{cut} \quad \dfrac{\Gamma, (\Delta, A^-); [A^-] \vdash c}{\Gamma, (\Delta, A^-) \vdash c}$$

$$\rightsquigarrow$$

Cut principles

$$\frac{\Delta \Vdash A^+ \quad \Gamma \vdash \Delta}{\Gamma \vdash [A^+]} \quad \text{cut} \quad \frac{\forall(\Delta \Vdash A^+) : \quad \Gamma, \Delta \vdash \boldsymbol{c}}{\Gamma; A^- \vdash \boldsymbol{c}}$$

$$\rightsquigarrow$$

$$\Gamma \vdash \Delta \quad \text{cut} \quad \Gamma, \Delta \vdash \boldsymbol{c}$$

. . . .

$$\frac{\Gamma \vdash \Delta \quad \Gamma \vdash A^-}{\Gamma \vdash \Delta, A^-} \quad \text{cut} \quad \frac{\Gamma, (\Delta, A^-); [A^-] \vdash \boldsymbol{c}}{\Gamma, (\Delta, A^-) \vdash \boldsymbol{c}}$$

$$\rightsquigarrow$$

$$\Gamma \vdash A^- \quad \text{cut} \quad \Gamma; [A^-] \vdash \boldsymbol{c}$$

Modularity

The proofs of identity and cut are completely generic!
(No connective mentioned.)

Though not necessarily terminating. . .

- $\Delta \Vdash A^+$ and $\Delta; A^- \Vdash \mathcal{C}$ induce a subformula ordering
- Ask whether it is well-founded

Modularity is nice:

- Simple cut-elimination for powerful logics (cf. Buchholz)
- Can add new connectives without worrying too much. . .

From AHOS to HOAS

(joint work with Dan Licata and Bob Harper)

A reflection on ∇

Definitional reflection defines propositional constants by rules:

$$\frac{A_1 \quad \dots \quad A_n}{P}$$

Notation: $P \Leftarrow A_1 \Leftarrow \dots \Leftarrow A_n$

$\nabla x.A$ introduces a new, scoped term constant

Idea: can we introduce new, scoped *rules*?

(C-H: scoped *constructor patterns*)

Definitional variation

Pattern-typing indexed by signature

$$\boxed{\Psi; \Delta \Vdash A^+ \qquad \Psi; \Delta; A^- \Vdash \boldsymbol{c}}$$

Pattern-typing for definitions:

$$\frac{P \Leftarrow A_1^+ \Leftarrow \ldots \Leftarrow A_n^+ \in \Psi \qquad \Psi; \Delta_1 \Vdash A_1^+ \quad \ldots \quad \Psi; \Delta_n \Vdash A_n^+}{\Psi; \Delta_1, \ldots, \Delta_n \Vdash P}$$

Positive $R \Rightarrow A^+$, negative $R \curlywedge A^-$

$$\frac{\Psi, R; \Delta \Vdash A^+}{\Psi; \Delta \Vdash R \Rightarrow A^+} \qquad \frac{\Psi, R; \Delta; A^- \Vdash \boldsymbol{c}}{\Psi; \Delta; R \curlywedge A^- \Vdash \boldsymbol{c}}$$

(C-H: higher-order patterns $\lambda u.p$ and $[u]; d$)

Definitional variation (cont.)

Make (non-atomic) hypotheses & conclusions *contextual*

Hypothesis	h	$::= X^{+} \mid \langle\Psi\rangle\, A^{-}$
Conclusion	c	$::= X^{-} \mid \langle\Psi\rangle\, A^{+}$

Reuse the same focusing rules! (And proofs of identity & cut.)

System implemented in Agda2

- Uses de Bruijn indices to implement higher-order patterns
- Ψ does not always obey substitution (or weakening)
- But generic substitution for LF fragment
- For details, see tech report

Conclusions

Focusing is awesome!

Some directions & questions. . .

Proof theory

- More refined analysis of cut-elimination
- Second-order quantifiers (uniformity), dependent types

Programming languages

- Intersection & union types, dependent types
- Multiple polarities for notions of effects (Filinski, McBride)

Proof search

- Are Ω-rules useful?
- Proof search with effects?