• Part I Some Linear Algebra
  – Might be useful to HW2 and later courses
Basics

- We will use lower case letters for vectors, and upper case letters from matrixes. There elements are referred by $x_i, A_{i,j}$. Refer A’s column vectors as $A_j$
- $AB$
  - still remember what is matrix multiplication?
- $A=A^T$
  - transpose and symmetric matrix
- $a \cdot b = a^Tb, \ a \cdot b = |a|_2$
  - inner product, vectors are also matrixes
- $AA^{-1}=I$
  - Inverse and the identity matrix
- $tr(A)=\text{diag}(A)^T1$
  - trace, and the diagonal of a matrix
Basis and Space

- $\text{span}(x_1, x_2, x_3) = \{a_1x_1 + a_2x_2 + a_3x_3 \mid a_i \in \mathbb{R}\}$
  - the span of a set of vectors is a subspace in the $\mathbb{R}^d$ space, assuming $x_i$ are vectors in $\mathbb{R}^d$ space

- $\text{col}(A) = \{x \mid x = Ab\}$
  - $A$’s column space is the span of $A$’s column vectors

- $\text{row}(A) = \{x \mid x = A^T b\}$
  - $A$’s rows space is the span of $A$’s rows vectors

- Basis
  - A basis $B$ of a space $V$ is a linearly independent subset of $V$ that spans (or generates) $V$

- $e_i = (0, 0, \ldots, 1, \ldots, 0)$
  - the standard basis
Unitary Matrix

- If $a \cdot b = 0$, $|a|_2 \neq 0$, $|b|_2 \neq 0$,
  - then $a$ and $b$ are orthogonal

- If $n$-by-$n$ matrix $A$, $A^T A = I$
  - then $A$ is an unitary matrix
  - $|A_i|_2 = 1$ for any $i$
  - and $A_i \cdot A_j = 0$, for $i \neq j$

- If $A$ is unitary, then $A^T$ is also unitary
Rank of a Matrix

• rank(A) (the rank of a m-by-n matrix A) is
  – The maximal number of linearly independent columns
  – The maximal number of linearly independent rows
  – The dimension of col(A)
  – The dimension of row(A)

• If A is n by m, then
  – rank(A) <= \text{min}(m,n)
  – If n=\text{rank}(A), then A has full row rank
  – If m=\text{rank}(A), then A has full column rank
Singular Value Decomposition (SVD)

• Any matrix $A$ can be decomposed as $A = UDV^T$, where
  – where $D$ is diagonal, with $d = \text{rank}(A)$ non-zero elements
  – $U$ and $V$ are unitary matrices
  – The first $d$ rows of $U$ are orthogonal basis for $\text{col}(A)$
  – The first $d$ rows of $V$ are orthogonal basis for $\text{row}(A)$

• Re-interpreting $Ab$
  – Decompose $b$ by $V$ basis
  – Scale it by $\text{diag}(D)$
  – Then map it to the space spanned by $U$ basis
Eigen Value Decomposition

• Any symmetric matrix $A$ can be decompose as $A=UDU^T$, where
  – where $D$ is diagonal, with $d=\text{rank}(A)$ non-zero elements
  – The first $d$ rows of $U$ are orthogonal basis for $\text{col}(A)=\text{row}(A)$

• Re-interpreting $Ab$
  – first stretch $b$ along the direction of $u_1$ by $d_1$ times
  – Then further stretch it along the direction of $u_2$ by $d_2$ times

[Diagram showing vector $b$ transforming through $u_1$ and $u_2$]
Inversing a Low Rank Covariance Matrix

- In many applications (e.g. linear regression, Gaussian model) we need to calculate the inverse of covariance matrix $X^TX + \lambda I$
  - where each row of $X$ is a data sample
  - $I$ is an identity matrix for regularization

- If the number of feature is huge (e.g. each sample is an image, #sample $n << \#feature d$)
  - then $X$ is an very wide and short matrix
  - inversing $X^TX + \lambda I$ becomes an problem
    - the complexity of matrix inversion is generally $O(n^3)$
    - Matlab can comfortably solve matrix with $d=$thousand, but not much more than that
Inversing a Low Rank Covariance Matrix

• With the help of SVD, we actually don’t need to explicitly inverse $X^TX + \lambda I$
  – Decompose $X = UDV^T$
  – Then $X^TX + \lambda I = VD U^T U D V^T + \lambda I = V(D^2 + \lambda I) V^T$

  – Since $V(D^2 + \lambda I) V^T V(D^2 + \lambda I)^{-1} V^T = I$
  – We know that $(X^TX + \lambda I)^{-1} = V(D^2 + \lambda I)^{-1} V^T$
    • Inversing a diagonal matrix $D^2 + \lambda I$ is trivial
• Part II Matlab
  – Might be useful to HW2
Matlab

• Very easy to do matrix manipulation in Matlab

• Available for installs by contacting help+@cs.cmu.edu

• If this is your first time using Matlab
  – Strongly suggest you go through the “Getting Started” part of Matlab help
  – Many useful basic syntax
Making Matrix

- \( A=[1 \ 2 \ 3; \ 4 \ 5 \ 6; \ 7 \ 8 \ 9] \)
- \( A=\text{ones}(m,n) \)
- \( A=\text{zeros}(m,n) \)
- \( A=\text{eye}(n) \)
- \( A=\text{diag}([1 \ 2 \ 3]) \)
Referencing Matrix

- $A(i,j)$
  - reference a single element
- $A(i,:), A(:,j)$
  - reference a whole row/column
- $b=1:3:100; \ A(b,:)$
  - using vector as index
- $b=\text{diag}(A)$
  - reference the diagonal vector
Matrix Manipulation

• $C = A'$;
  – transpose

• $C = A + B; \ D = A \times B;$

• $D = A^3$
  – Equal to $A \times A \times A$

• $x = A \backslash b; \ x = b / A$
  – multiply the inverse of a matrix

• $D = A \times B; \ D = A \div B; \ D = A \backslash B; \ D = A \times \times 3;$
  – Point wise multiplication/division/power
Matrix Decomposition

- $[U, S, V] = \text{svd}(X)$
- $[V, D] = \text{eig}(A)$
• The End
• Thanks