Learning Combinatorial Functions

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2-Minute Version

This talk: learning combinatorial functions from data.

Relevant to many AI, ML, multi-agent settings. E.g.:

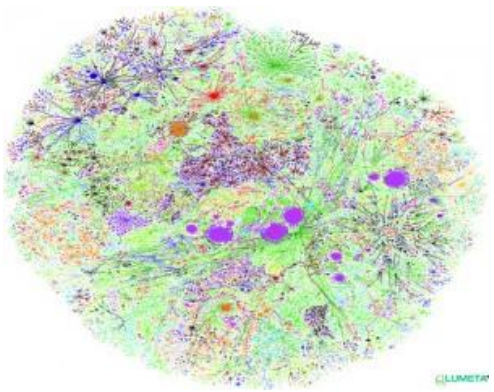
- Learning valuation functions of users for combinatorial auctions

- Learning the influence function in information diffusion networks, e.g., social networks

- Learning value function of various coalitions of agents in cooperative game theory

2-Minute Version

In large multi-agent systems (many users, huge networks) cannot assume these core objects (e.g., valuation functions, influence functions, values of coalitions) are known in advance.

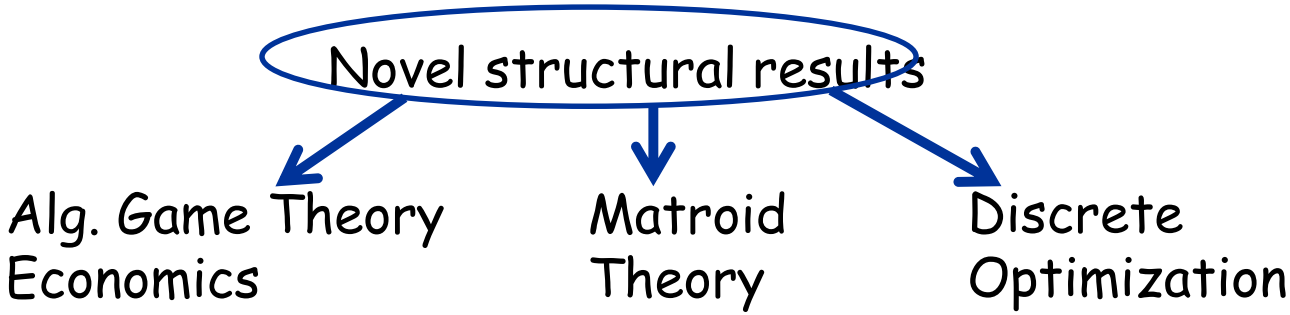


Techniques for learning them, better decision making.

2-Minute Version

This talk: learning combinatorial functions from data.

- General learnability results in a statistical setting.
- Surprising lower bounds for learning submodular functions showing unexpected structure.



- Much **better upper bounds** for combinatorial fns with more structure, coming from social networks & alg. game theory.
- Application for learning influence fnc in diffusion networks.

Structure of the talk

- Standard formulation for learning from data.

- Combinatorial set functions (submodular functions, coverage functions, etc).

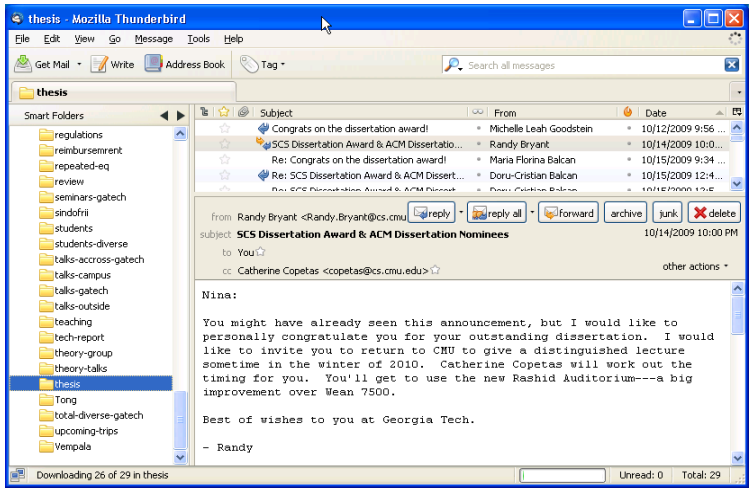
- Learning combinatorial functions.

With connections and applications to Algorithmic Game Theory, Economics, Social Networks.

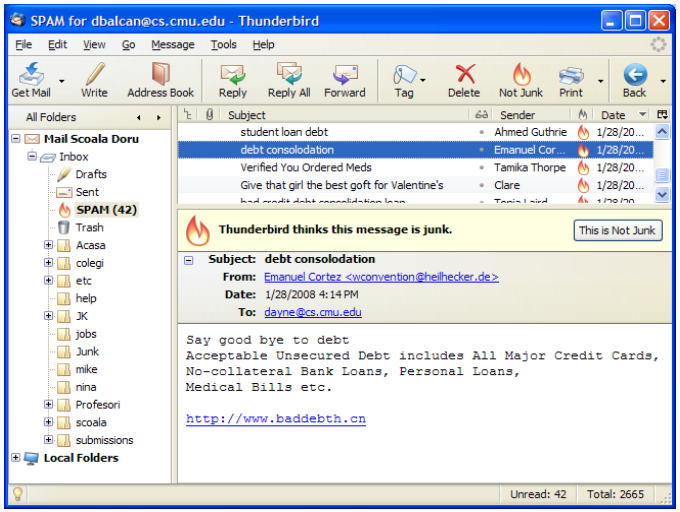
Supervised Machine Learning

- E.g., which emails are spam and which are important.

Not spam



spam



- E.g., will a news article be of interest to a user; will a car deal or sale would be of interest to a user.

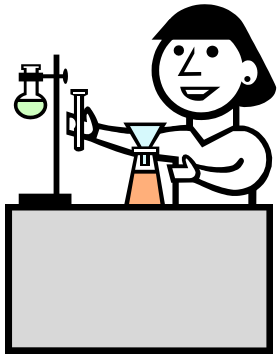
Statistical / PAC learning model

Data Source

Distribution D on X

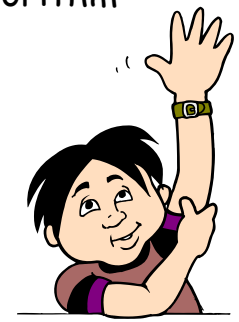
$(x_1,\ldots,x_k)$

Expert / Oracle



$f : X \to \{0,1\}$

Learning Algorithm

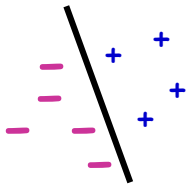


Labeled Examples

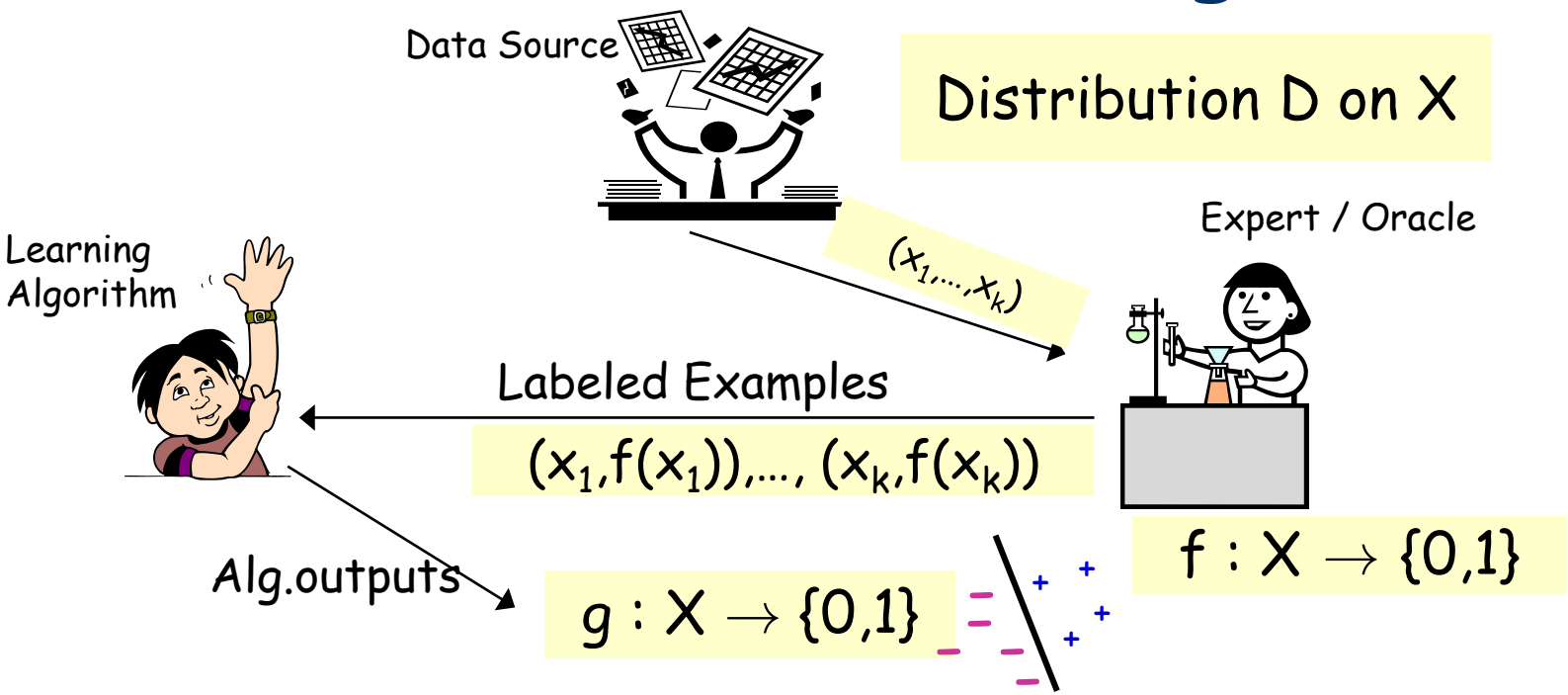
$(x_1,f(x_1)),\ldots,(x_k,f(x_k))$

Alg.outputs

$g : X \to \{0,1\}$



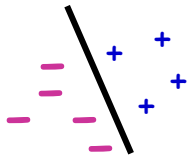
Statistical / PAC learning model



- Algo sees $(x_1,f(x_1)),\ldots,(x_k,f(x_k))$, x_i i.i.d. from D, produces g.
  - Want: with probability $\geq 1-\delta$ we have $\Pr_x[g(x)=f(x)] \geq 1-\epsilon$

Probably Approximately Correct (PAC)

Supervised Machine Learning

- Can learn important classes of Boolean functions:
  - linear separators 
  - kernelized linear separators (i.e., linear separators in implicit higher dim features spaces)
- Successful algorithms (e.g., Boosting, SVM).

This talk: how well we can learn important classes of combinatorial real-valued fns (e.g., submodular functions).

Structure of the talk

- Standard formulation for learning from data.

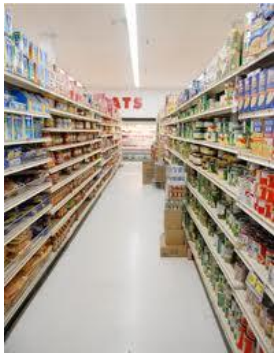
- Combinatorial set functions (submodular functions, coverage functions, etc).

- Learning combinatorial functions.

With connections and applications to Algorithmic Game Theory, Economics, Social Networks.

Combinatorial Set Functions

- Function over sets.
  - e.g., value on some set of items in a store.



- Formally, ground set $V=\{1,2, \ldots, n\}$.

set-functions $f : 2^V \rightarrow R$

This talk: focus on

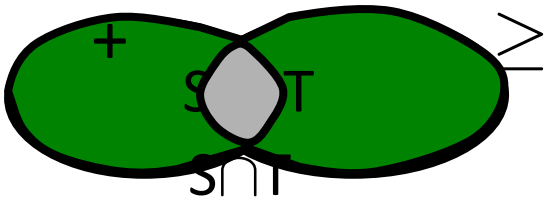
<u>Monotone</u>: $f(S) \leq f(T)\ ,\ \forall\ S \subseteq T$

<u>Non-negative</u>: $f(S) \geq 0, \forall\ S \subseteq V$

Submodular functions

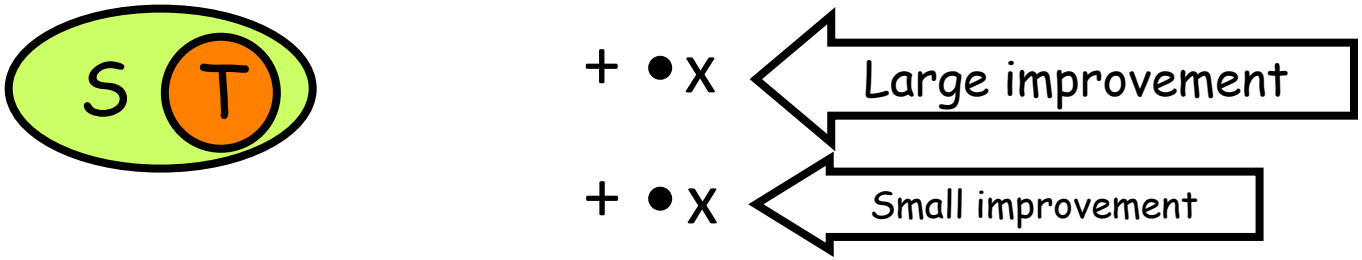
- $V=\{1,2, \ldots, n\}$; set-function $f : 2^V \rightarrow R$ submodular if

  For all $S,T \subseteq V$: $f(S)+f(T) \geq f(S \cap T)+f(S \cup T)$



- Equivalent decreasing marginal return:

  For $T \subseteq S$, $x \notin S$, $f(T \cup \{x\}) - f(T) \geq f(S \cup \{x\}) - f(S)$



Submodular functions

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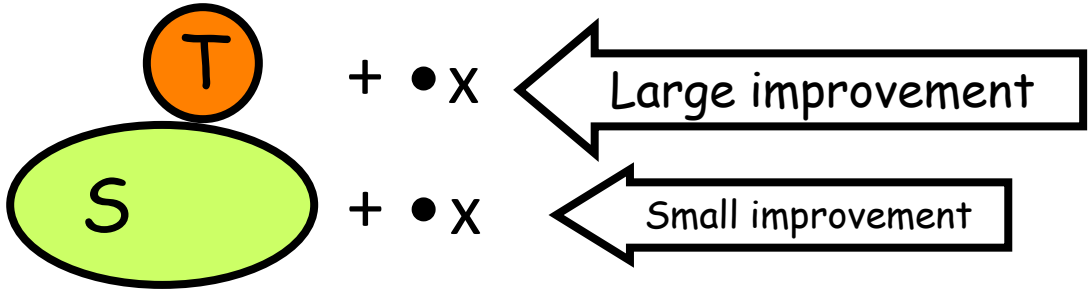
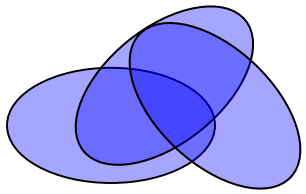


E.g.,

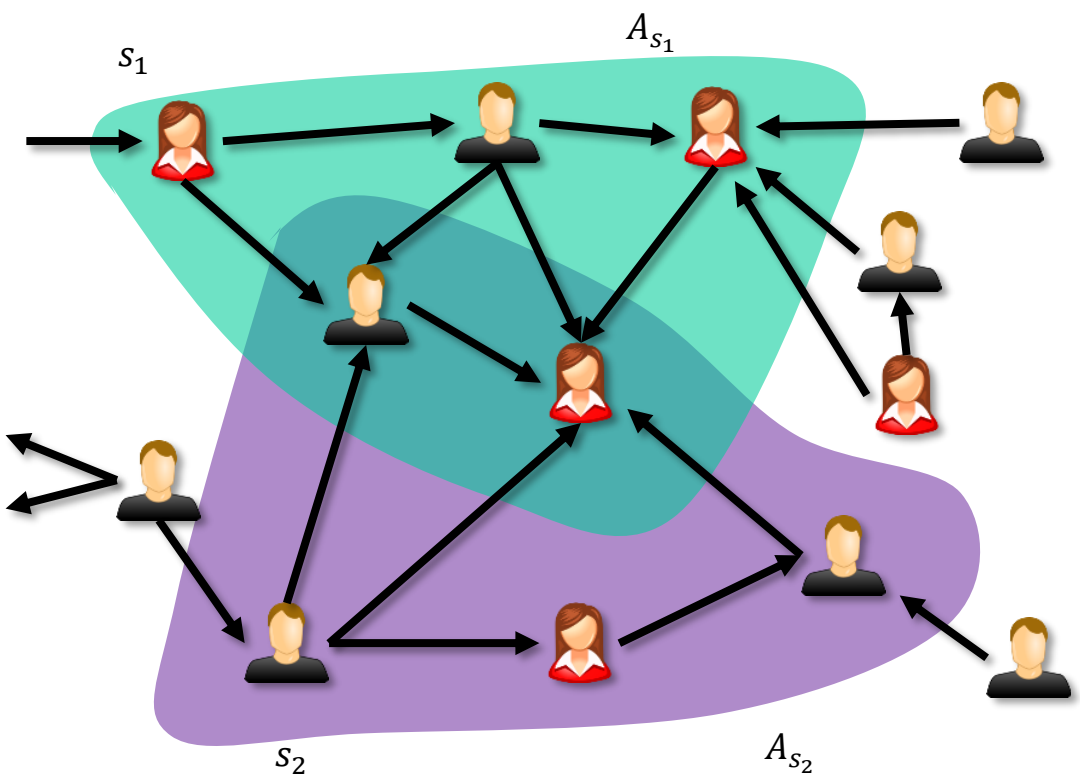
Coverage and Reachability Functions

- **Coverage function:** Let $A_1, \ldots, A_n$ be sets.
  For each $S \subseteq V$, let $f(S) = |\cup_{j \in S} A_j|$



- **Reachability function:** $f(S) = \#$ nodes reachable from S.

  E.g., in a network, A_s nodes reachable from s



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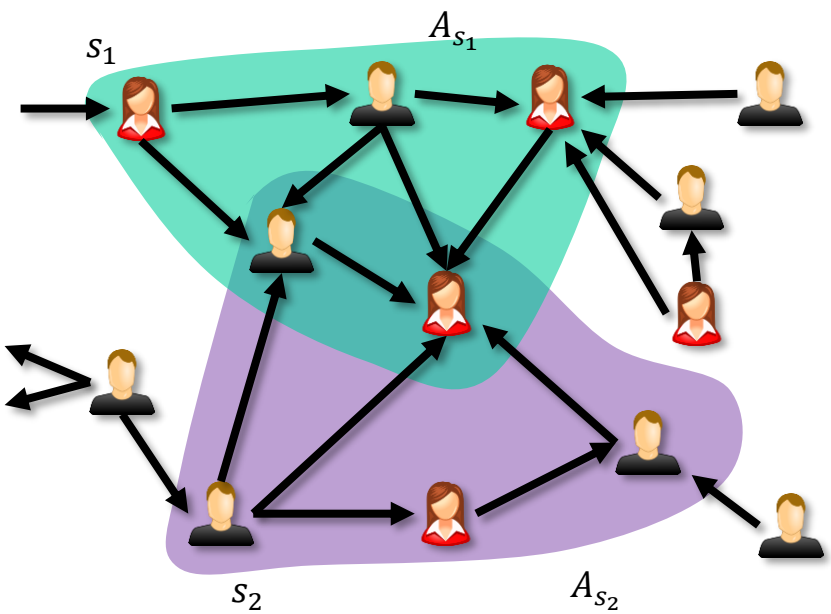


Diminishing Returns

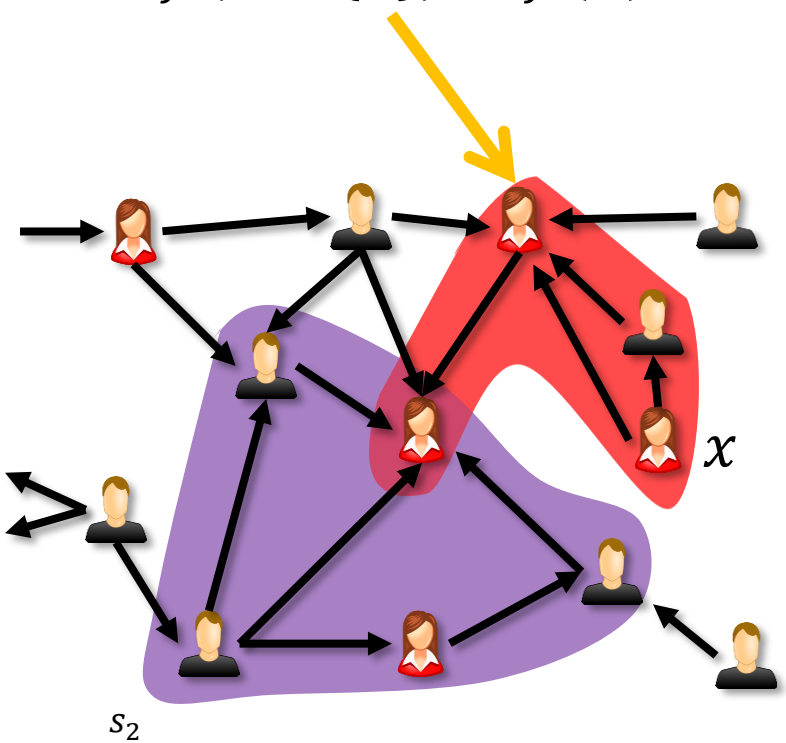
- Marginal value of x given S is # number of new nodes that x can reach, but cannot be reached from any of the nodes in S.
- $T \subset S, x \notin S$, more chance reach new nodes when adding x to T, than when adding x to S.

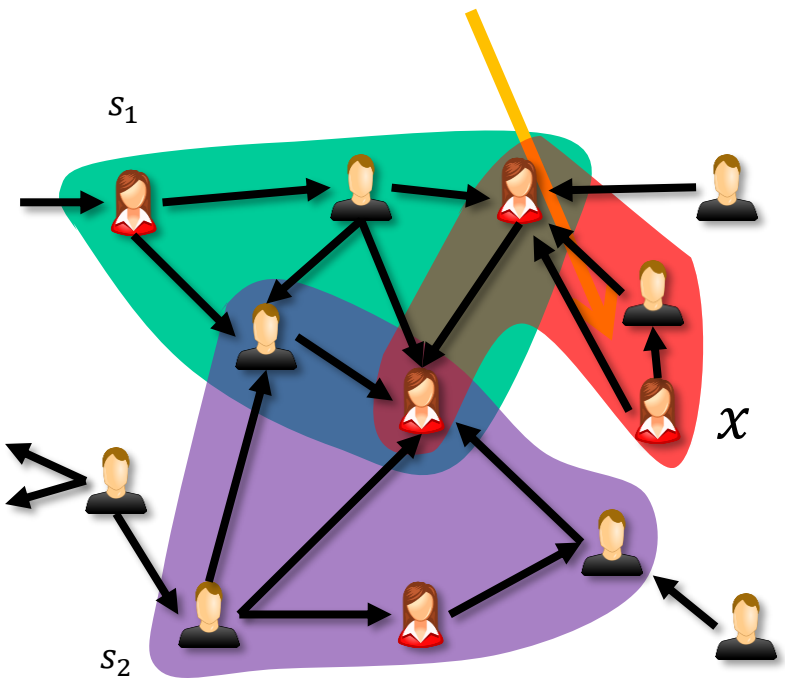
Reachability function is submodular

Marginal value of x = # new nodes reachable from x.

$T = \{s_2\}, \quad f(T) = 5$ $\qquad\qquad$ $S = \{s_1, s_2\}, \quad f(S) = 8$

$$\overset{3}{f(T \cup \{x\}) - f(T)} \geq \overset{2}{f(S \cup \{x\}) - f(S)}$$

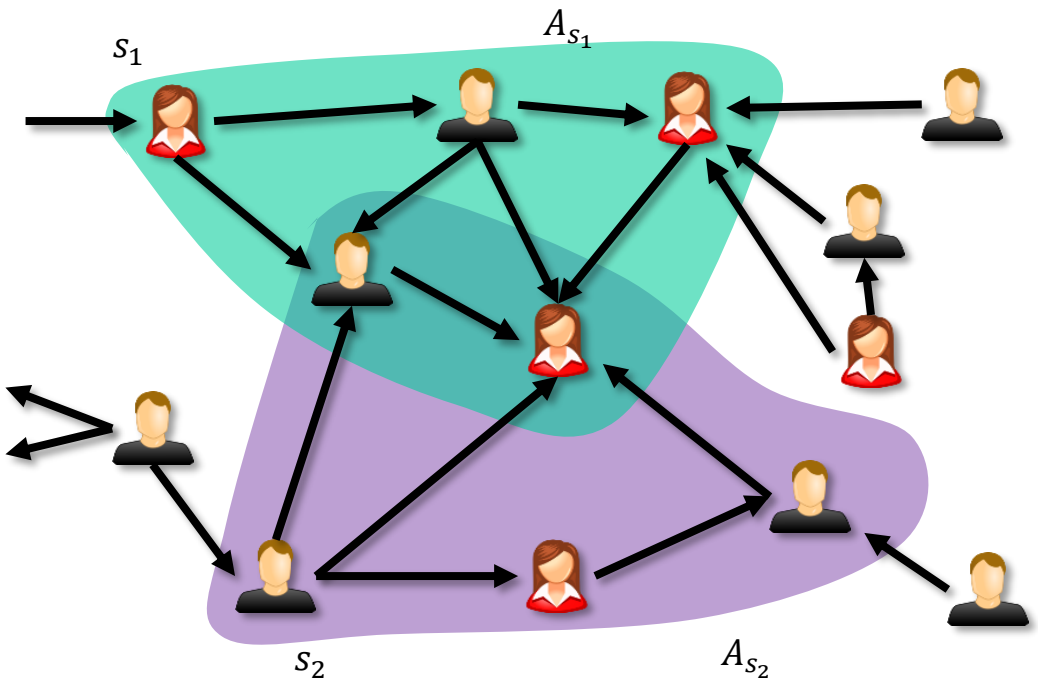




Probabilistic Reachability Functions

- Given a distribution over graphs

  $f(S) = E_G[\#\text{ reachable from } S|G]$ also submodular.



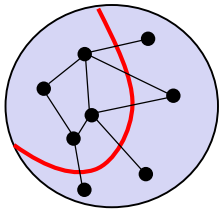
Submodular functions

More examples:

- **Concave Functions** Let $h : \mathbb{R} \to \mathbb{R}$ be concave.
  For each $S \subseteq V$, let $f(S) = h(|S|)$

- **Vector Spaces** Let $V=\{v_1,\ldots,v_n\}$, each $v_i \in R^n$.
  For each $S \subseteq V$, let $f(S) = \text{rank}(V[S])$

- **Cut Function in a Graph** Let $f(S)$ = # of edges between S and $V \setminus S$.



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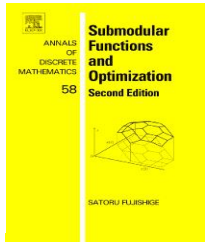
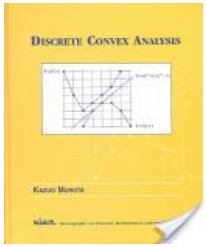
<u>Non-negative</u>: $f(S) \geq 0, \forall\ S \subseteq V$

Combinatorial Functions

- A lot of work on Optimization Problems, including optimization of Submodular Functions.

<u>Traditionally</u>: Optimization, operations research





<u>Most recently</u>

- Algorithmic Game Theory [Lehman-Lehman-Nisan'01], ...
- Machine Learning [Bilmes'03] [Guestrin-Krause'07], ...
- Social Networks [Kleinberg-Kempe-Tardos'03]

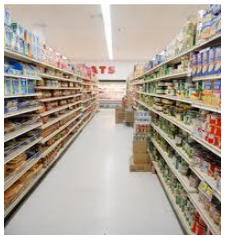
- **This talk**: learning them from data.

Learning Combinatorial Functions

Combinatorial fns such as submodular fns commonly used in Economics and AGT to model **valuation functions** of customers.

Supermarket chain

- V = all the items you sell.
- f(S) = valuation on set of items S.



Learning Combinatorial Functions

In online social networks and other networks:

- How much influence various sets of individuals/nodes have?

- If a given set of individuals are the initial adaptors of an idea or product, what is the expected number of followers?

Learning Combinatorial Functions

Influence Function in Social Networks

- V = set of nodes.
- f(S) = expected number of nodes S will influence.

f is a probabilistic reachability fnc in classic diffusion models (e.g., independent cascade model, random threshold model) [Kleinberg-Kempe-Tardos'03]

Past Work

Assume an explicit model on how info spreads ; use it to estimate the influence fnc.

Could be mis-specified.

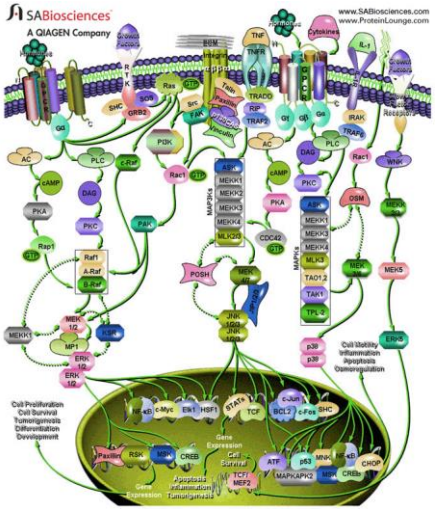
Our Work

Learn the influence function directly from data

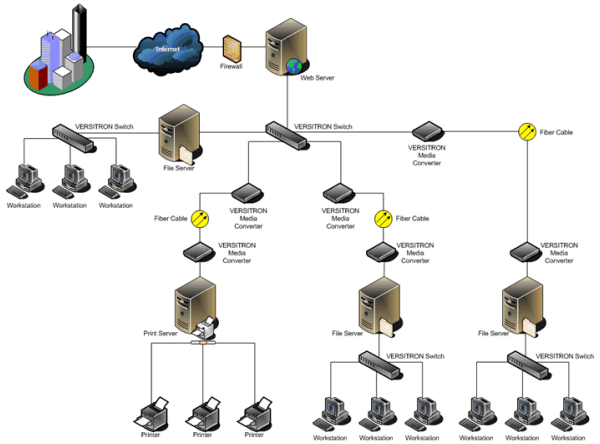
Learning Combinatorial Functions

Influence Function in Networks

epidemiology: influenza spread



biology:
gene expression cascade



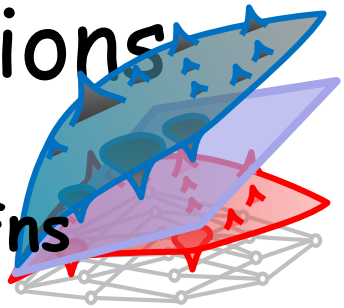
virus
report

Symantec.

TREND MICRO

cybersecurity:
computer virus spread

Learning Combinatorial Functions



General Learnability Results for Submodular Fns

- Upper & lower bounds on their intrinsic complexity.
  - Implications to Alg. Game Theory, Economics, Discrete Optimization, Matroid Theory.
- Highlights importance of beyond worst case analysis.

Better Results for Cases with More Structure

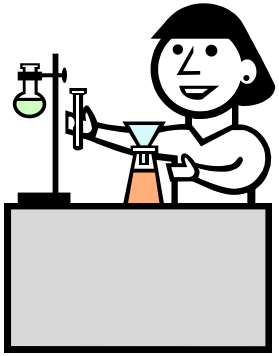
Large Scale Application to Social Networks

Statistical learning model

Data Source

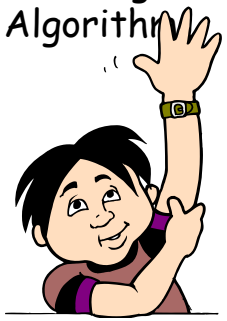
Distribution D on $2^{[n]}$

Expert / Oracle



$f : 2^{[n]} \rightarrow R_+$

Learning Algorithm



Labeled Examples

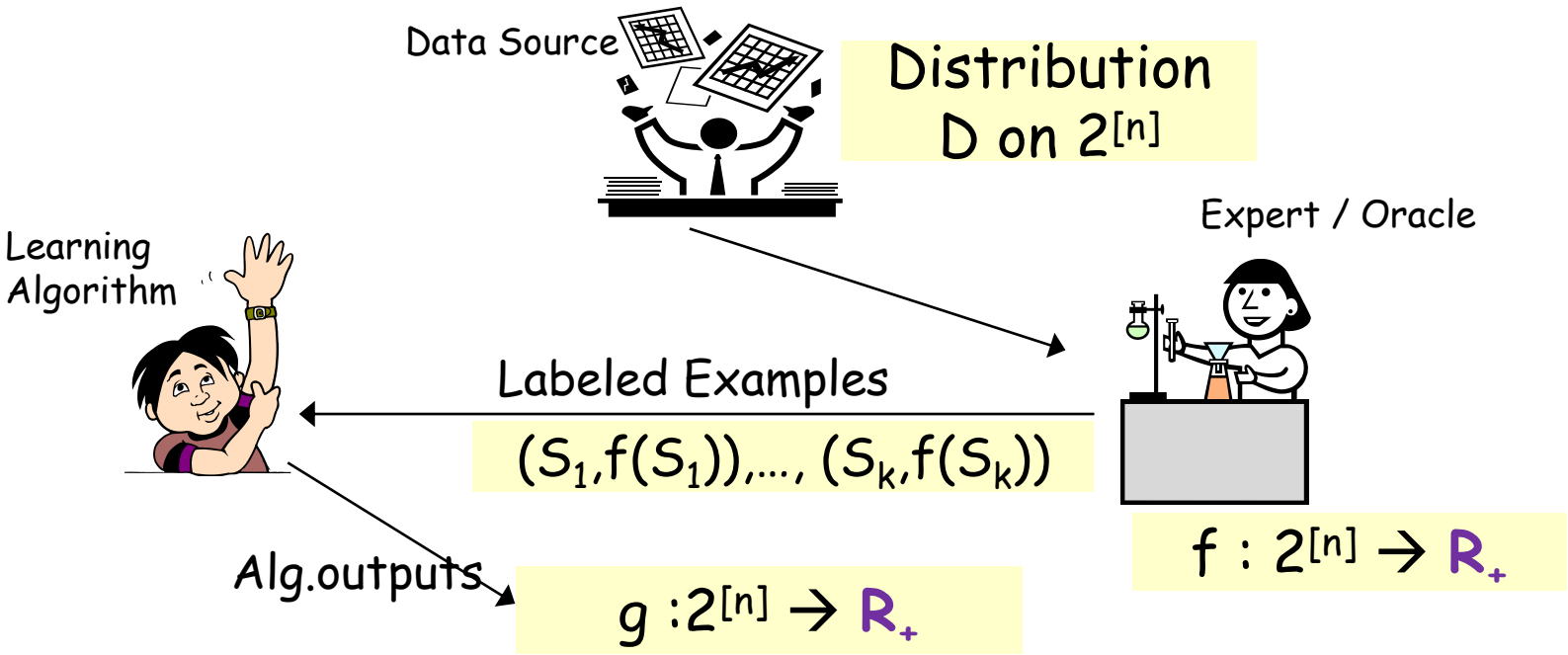
$(S_1, f(S_1)), \ldots, (S_k, f(S_k))$

Alg.outputs

$g : 2^{[n]} \rightarrow R_+$

PMAC model for learning real valued functions

[Balcan&Harvey, STOC 2011 & Nectar Track, ECML-PKDD 2012, SICOMP 2018]

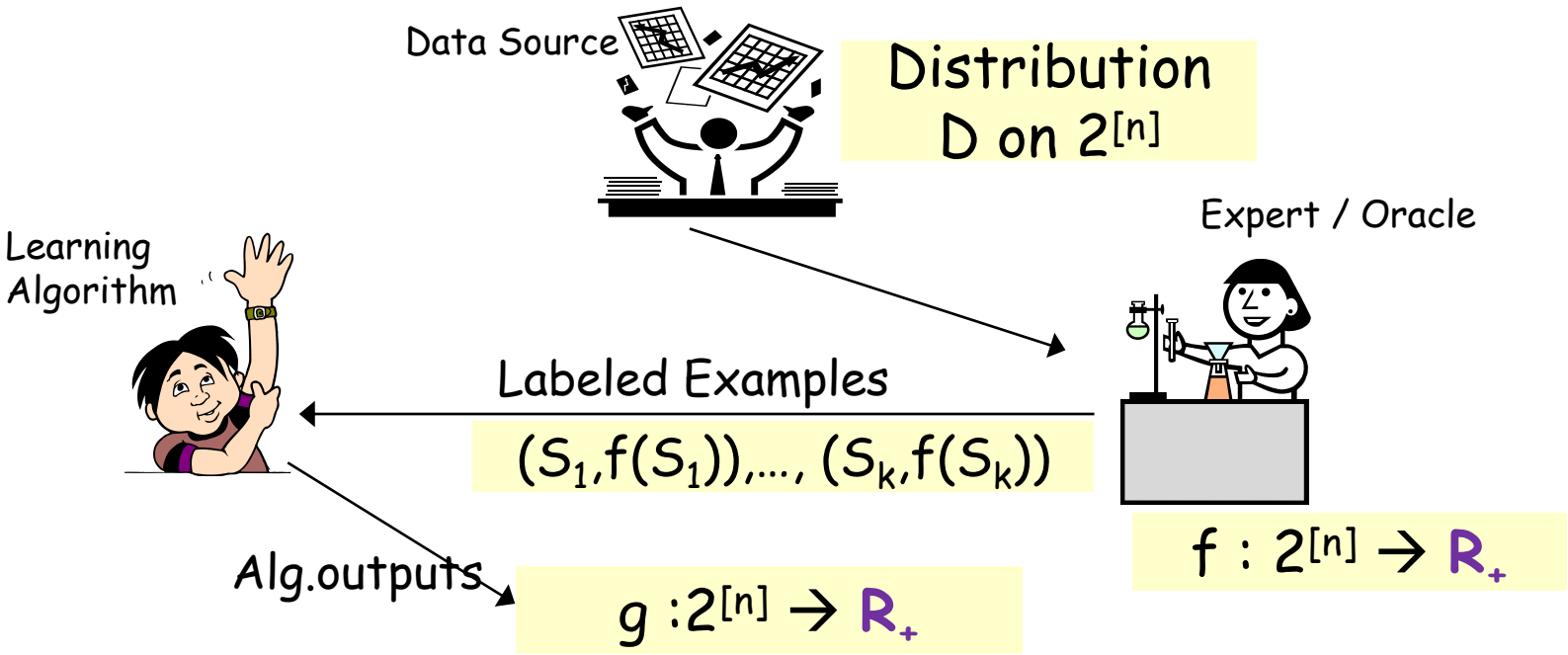


- Algo sees $(S_1,f(S_1)),\ldots,(S_k,f(S_k))$, S_i i.i.d. from D, produces g.
- With probability $\geq 1-\delta$ we have $\Pr_S[g(S) \leq f(S) \leq \alpha\, g(S)] \geq 1-\epsilon$

Probably Mostly Approximately Correct

PMAC model for learning real valued functions

[Balcan&Harvey, STOC 2011 & Nectar Track, ECML-PKDD 2012, SICOMP 2018]



- Algo sees $(S_1,f(S_1)),\ldots,(S_k,f(S_k))$, S_i i.i.d. from D, produces g.
- With probability $\geq 1-\delta$ we have $\Pr_S[g(S) \leq f(S) \leq \alpha\, g(S)] \geq 1-\epsilon$

$\alpha = 1$, recover PAC model.

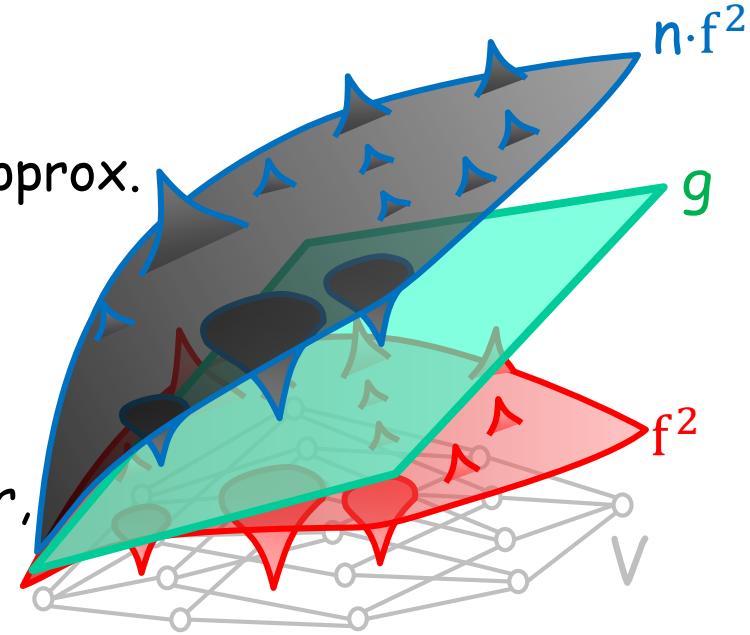
A General Upper Bound

Theorem: Can PMAC-learn submodular functions in poly time and poly samples with an approx. factor $\alpha = O(\sqrt{n})$.

Key Idea:

- Target f submodular, then f^2 can be approx. within n by a **linear** function.

$$f^2(S) \leq g(S) \leq n\, f^2(S), \forall S$$

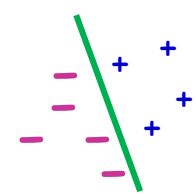


- Reduction to learning a linear separator, over an n+1 dimensional feature space.

we have a feature for each item in the ground set and one additional feature related to the value of the function

Given $(S_i, f(S_i)), i = 1, \ldots, m$, $((\chi(S_i), f^2(S)), +)$ and $((\chi(S_i), n\, f^2(S)), -)$

are linearly separable in R^{n+1}.



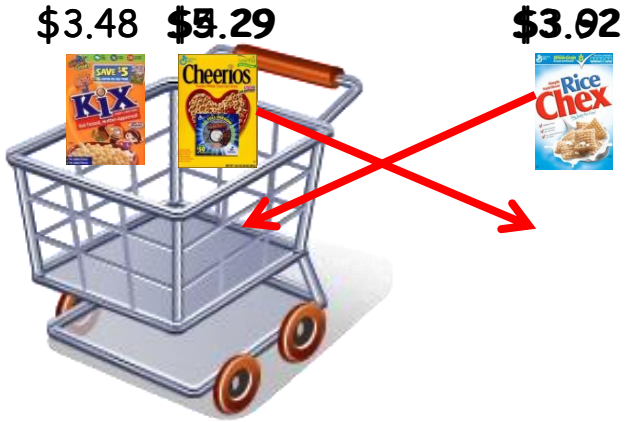
A General Lower Bound

Theorem: No algo can PMAC learn the class of submodular fns with approx. factor $\tilde{o}(n^{1/3})$.

- Even if value queries allowed; even for rank fns of matroids.

Corollary: Gross substitute fns do **not** have a concise, approximate representation.

- f satisfies gross substitutes if raising prices on some items does not remove any items from optimal bundle whose price did not change.



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Surprising answer to open question in AGT & Economics of

Paul Milgrom

Noam Nisan

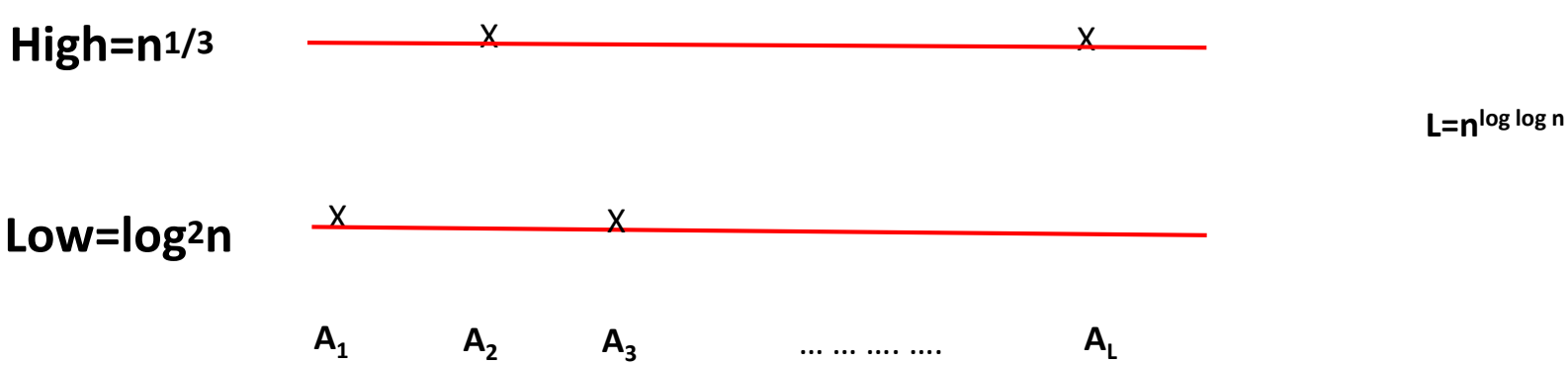
A General Lower Bound

Theorem

No algorithm can PMAC learn the class of non-neg., monotone, submodular fns with an approx. factor $\tilde{o}(n^{1/3})$.

Idea:

A new hard family of matroid rank functions with extremal properties.



Partition Matroids

$A_1, A_2, \ldots, A_k \subseteq V=\{1,2, \ldots, n\}$, all disjoint; $u_i \leq |A_i|-1$

$Ind=\{I: |I \cap A_j| \leq u_j$, for all $j\}$

Then (V, Ind) is a matroid.

If sets A_i are not disjoint, then (V,Ind) might not be a matroid.

- E.g., $n=5$, $A_1=\{1,2,3\}$, $A_2=\{3,4,5\}$, $u_1 = u_2=2$.
- $\{1,2,4,5\}$ and $\{2,3,4\}$ both maximal sets in Ind; do not have the same cardinality.

Almost partition matroids

$k=2$, $A_1, A_2 \subseteq V$ (not necessarily disjoint); $u_i \leq |A_i|-1$

$\mathrm{Ind}=\{I: |I \cap A_j| \leq u_j\ ,\ |I \cap (A_1 \cup A_2)| \leq u_1 + u_2 - |A_1 \cap A_2|\}$

Then (V,Ind) is a matroid.

Almost partition matroids

More generally

$A_1, A_2, \ldots, A_k \subseteq V=\{1,2, \ldots, n\}$, $u_i \leq |A_i|-1$; $f : 2^{[k]} \to Z$

=<0

$$f(J)= \sum_{j \in J} u_j + |A(J)| - \sum_{j \in J} |A_j|, \forall J \subseteq [k]$$

$Ind= \{ I: |I \cap A(J)| \leq f(J), \forall J \subseteq [k] \}$

Then (V, Ind) is a matroid (if nonempty).

Rewrite f, $f(J)=|A(J)|-\sum_{j \in J}(|A_j| - u_j), \forall J \subseteq [k]$

Almost partition matroids

More generally $f : 2^{[k]} \to Z$

$$f(J)=|A(J)|-\sum_{j \in J}(|A_j| - u_j), \forall J \subseteq [k]$$

$$\text{Ind}= \{ I: |I \cap A(J)| \leq f(J), \forall J \subseteq [k] \}$$

Then (V, Ind) is a matroid (if nonempty).

$f : 2^{[k]} \to Z, f(J)=|A(J)|-\sum_{j \in J}(|A_j| -u_j), \forall J \subseteq [k]; u_i \leq |A_i|-1$

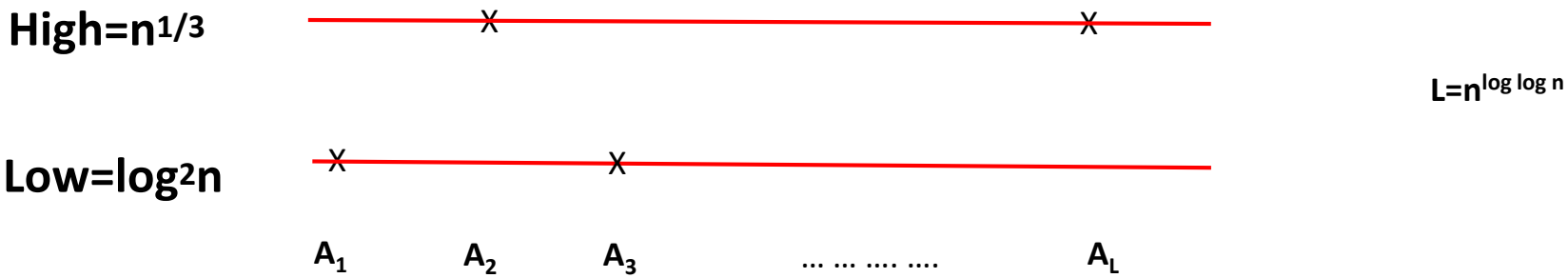
Note: This requires $k \leq n$ (for $k > n$, f becomes negative)

More tricks to allow $k=n^{\log \log n}$.

Learning submodular valuations

Theorem

No algorithm can PMAC learn the class of non-neg., monotone, submodular fns with an approx. factor $\tilde{o}(n^{1/3})$.



Worst Case Analysis ☺

Moral: Exploit Additional Structure

- Product distribution.
  [Balcan-Harvey,STOC'11 & SICOMP'18] [Feldman-Vondrak,FOCS'13]
- Bounded Curvature (i.e., almost linear)
  [Iyer-Jegelka-Bilmes, NIPS'13]

- Learning valuation fns from AGT and Economics.
  [Balcan-Constantin-Iwata-Wang, COLT '12]
  [Badanidiyuru-Dobzinski-Fu- Kleinberg-Nisan-Roughgarden, SODA'12]
- Learning influence fns in information diffusion networks
  [Du, Liang, Balcan, Song, ICML'14; NIPS'14]

- Learning values of coalitions in cooperative game theory
  [Balcan, Procacia, Zick, IJCAI'15]

Learning Valuation Functions

- Target dependent learnability for classes of valuation fns have a succinct description.

[Balcan-Constantin-Iwata-Wang, COLT 2012]

Well-studied subclasses of subadditive valuations.

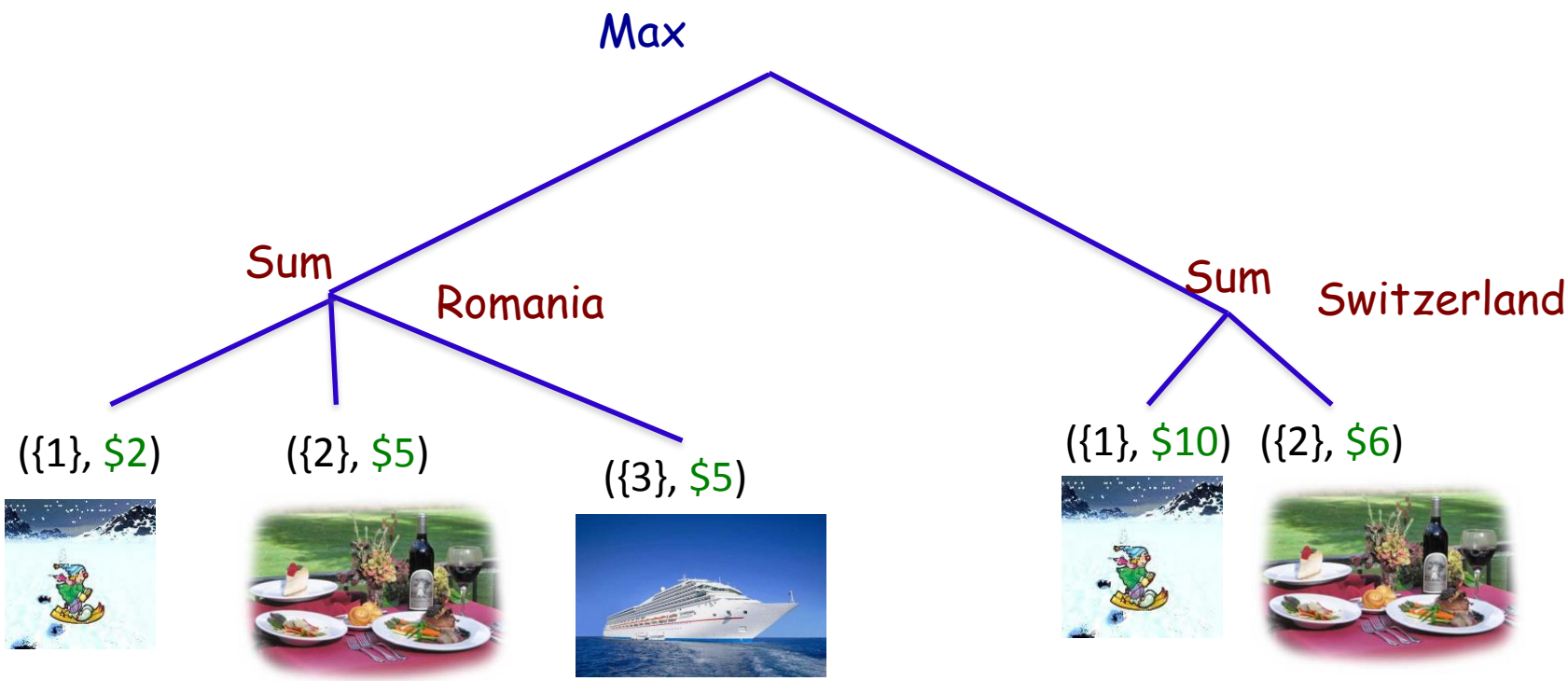
[Algorithmic game theory and Economics]

Additive $\subseteq$ OXS $\subseteq$ Submodular $\subseteq$ XOS $\subseteq$ Subadditive

[Sandholm'99] [Lehman-Lehman-Nisan'01]

XOS valuations

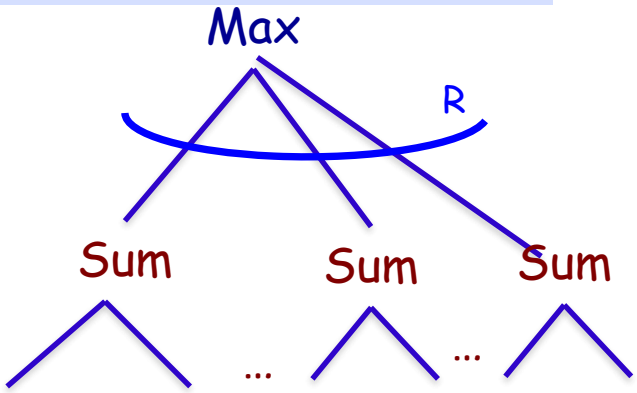
Functions that can be represented as a MAX of SUMs.



g({1,2}) = $16 g({2,3}) = $10 g({1,2,3}) = $16

Target dependent Upper Bound for XOS

Theorem: (Polynomial number of Sum trees)
$O(R^{\epsilon})$ approximation in time $O(n^{1/\epsilon})$.



Main Idea:

- Target approx within $O(R^{\epsilon})$ by a linear function over $O(n^{1/\epsilon})$ feature space (one feature for each $n^{1/\epsilon}$-tuple of items).

- Reduction to learning a linear separator in a higher dim. feature space.

Highlights importance of considering the complexity of the target function.

Learning Influence Functions in Information Diffusion Networks

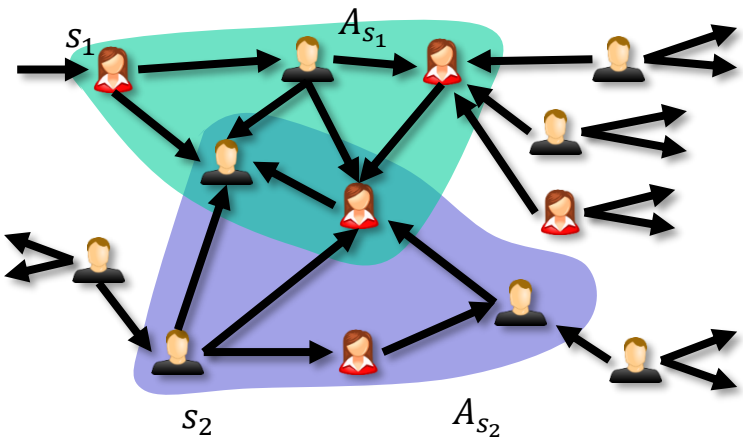
[Du, Liang, Balcan, Song, ICML 2014 , NIPS'14]

Influence Function in Networks

- V = set of nodes.
- f(S) = expected number of nodes S will influence.

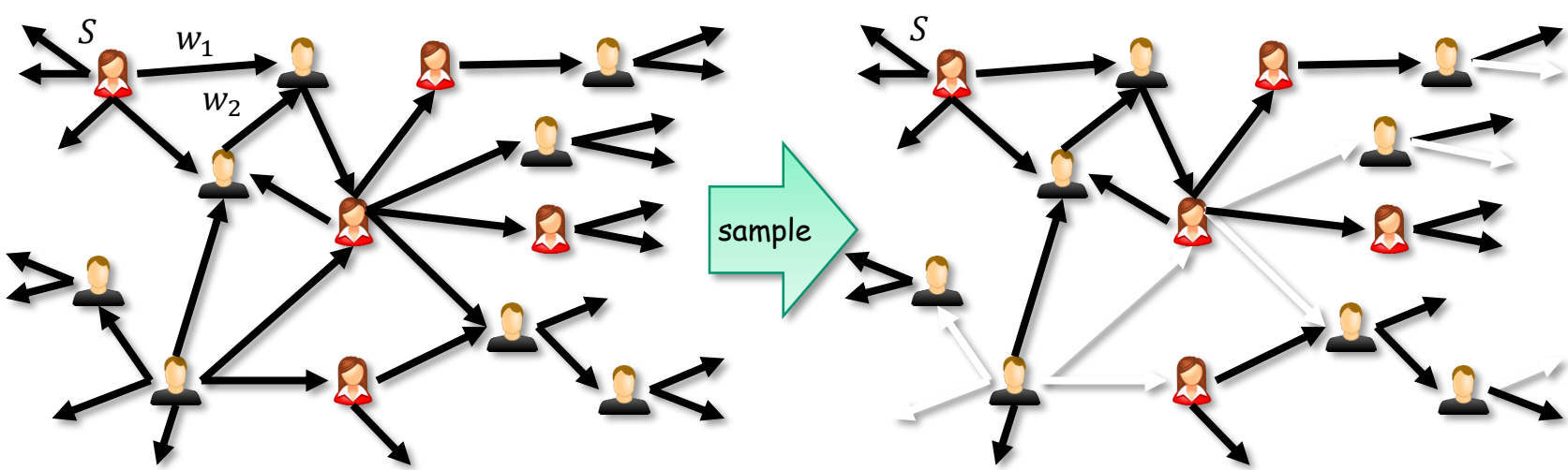
Fact: in classic diffusion models (discrete time independent cascade model/random threshold model, continuous time analogues, etc), the influence function is coverage function. [Kleinberg-Kempe-Tardos'03]

$f(S) = E_G[\#\text{ reachable from } S|G]$
probabilistic reachability fnc



Discrete-time independent cascade model

- Input: directed graph, each edge has a weight $w \in [0,1]$
- Cascade generative process for a source set S
  - presence of edge is sampled independently according to w
  - influenced nodes are those reachable from at least one of the source nodes in the resulting "live edge graph"
- Influence of S is expected number of nodes influenced under this random process



Learning Influence Functions in Information Diffusion Networks

[Du, Liang, Balcan, Song, ICML 2014 , NIPS'14]

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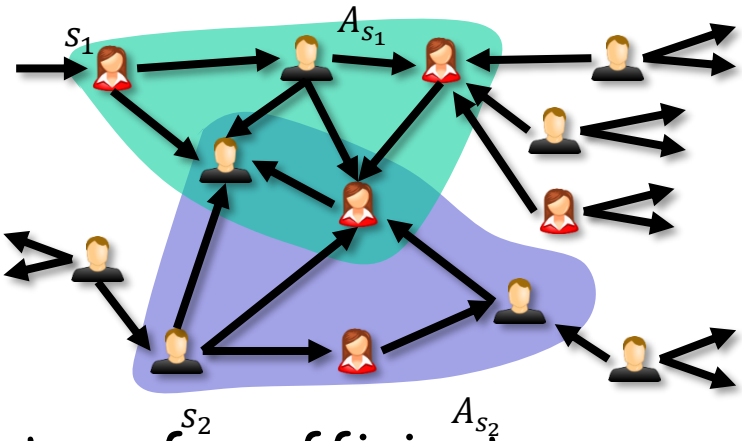
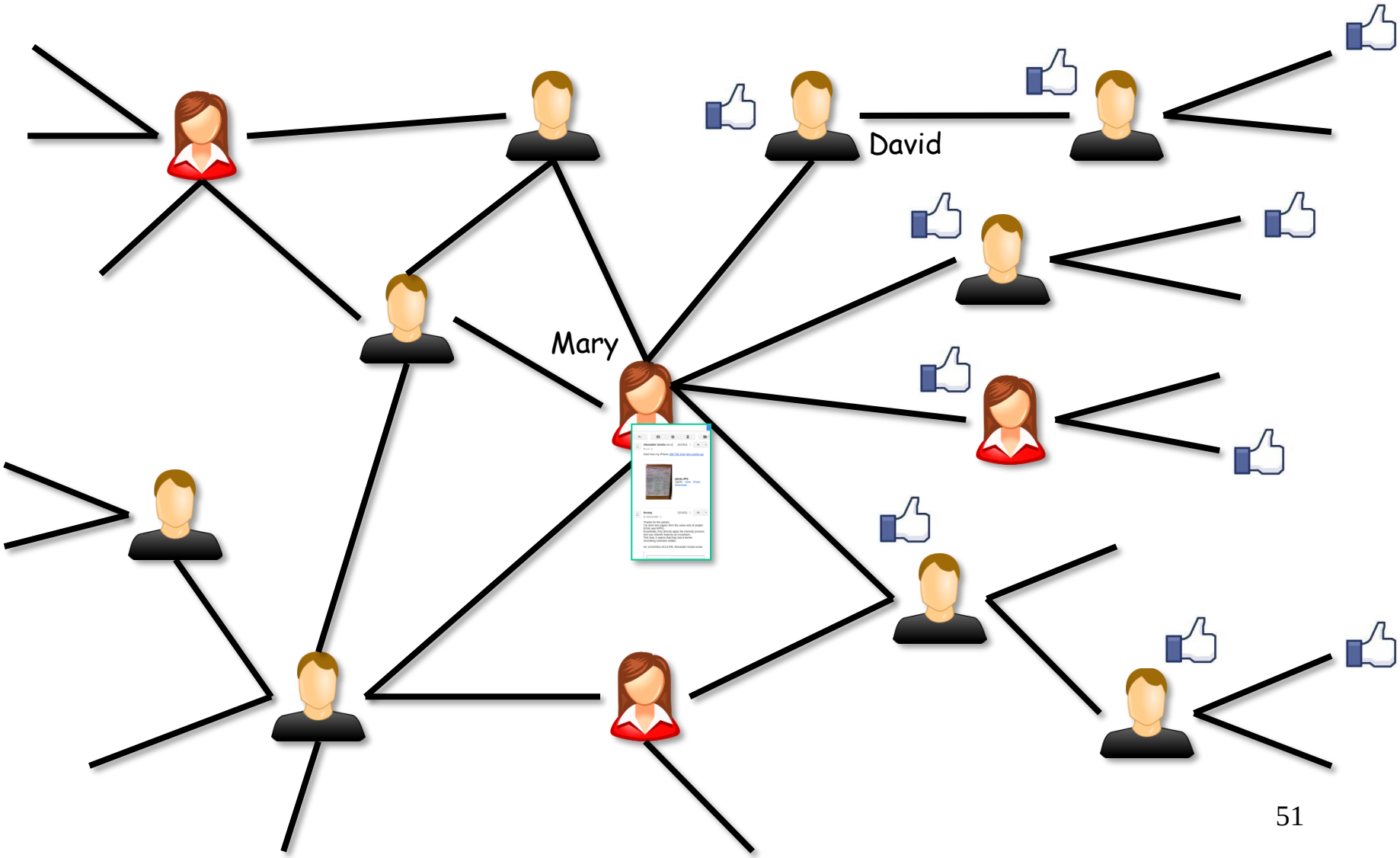


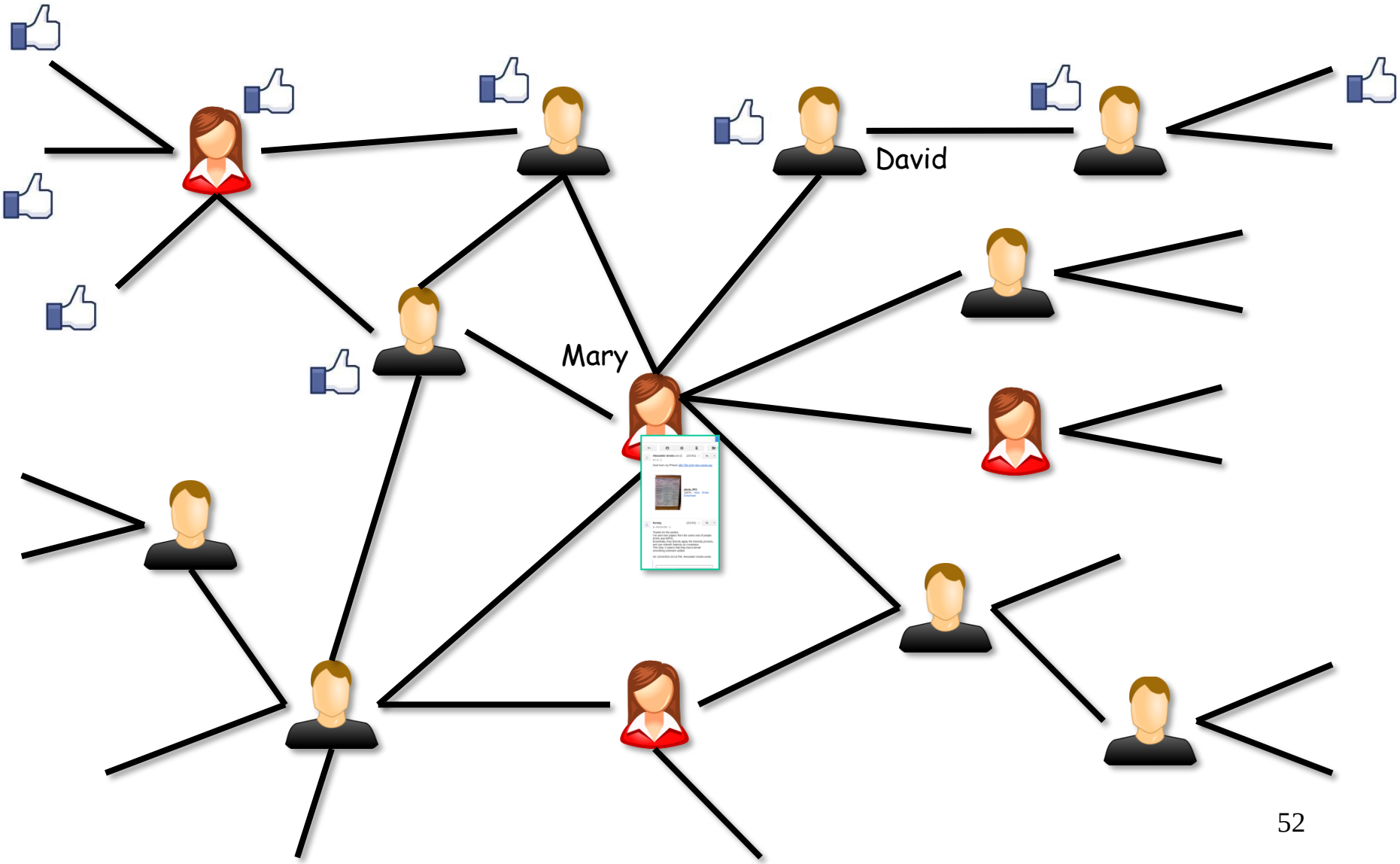
- **Note 1**: Do not know better guarantees for efficient algorithms if access only to value queries.

- **Note 2**: Do better theoretically and empirically, if have access to information diffusion traces or cascades.

Learning Influence Functions based on information propagation traces (cascades)



Another cascade



Learning the influence function

Input: past influence cascades $\{(S_1, I_1), (S_2, I_2), \dots, (S_m, I_m)\}$.

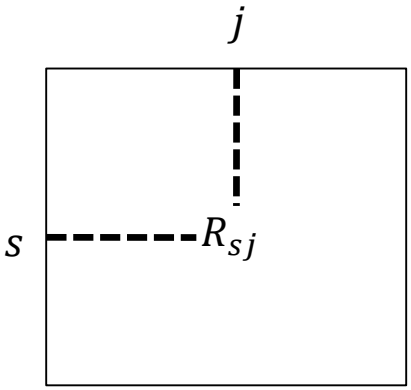
Goal: learn Influence function $f(S) = E[\#\text{influenced}(S)]$.

Assumption: $f(S)$ is a probabilistic coverage function.

I.e., there is a distribution p_R over reachability matrices R s.t.:

$$f(S) = E_{R \sim p_R}[\#\text{influenced}(S|R)]$$

$$|\{j: R_{sj} = 1 \text{ for some } s \in S\}|$$



$R_{sj} = 1$ if s can reach j,
$R_{sj} = 0$ otherwise.

Learning the influence function

Input: past influence cascades $\{(S_1, I_1), (S_2, I_2), \dots, (S_m, I_m)\}$.

Goal: learn Influence function $f(S) = E[\#\text{influenced}(S)]$.

Idea: $f(S) = \sum_j f_j(S)$, where $f_j(S) = \Pr_{R \sim p_R}(j \text{ is influenced by } S)$.

For each j, will learn $\hat{f}_j(S)$. Output $\sum_j \hat{f}_j(S)$.

Algorithm for learning f_j

Use "random kitchen sink" approach:

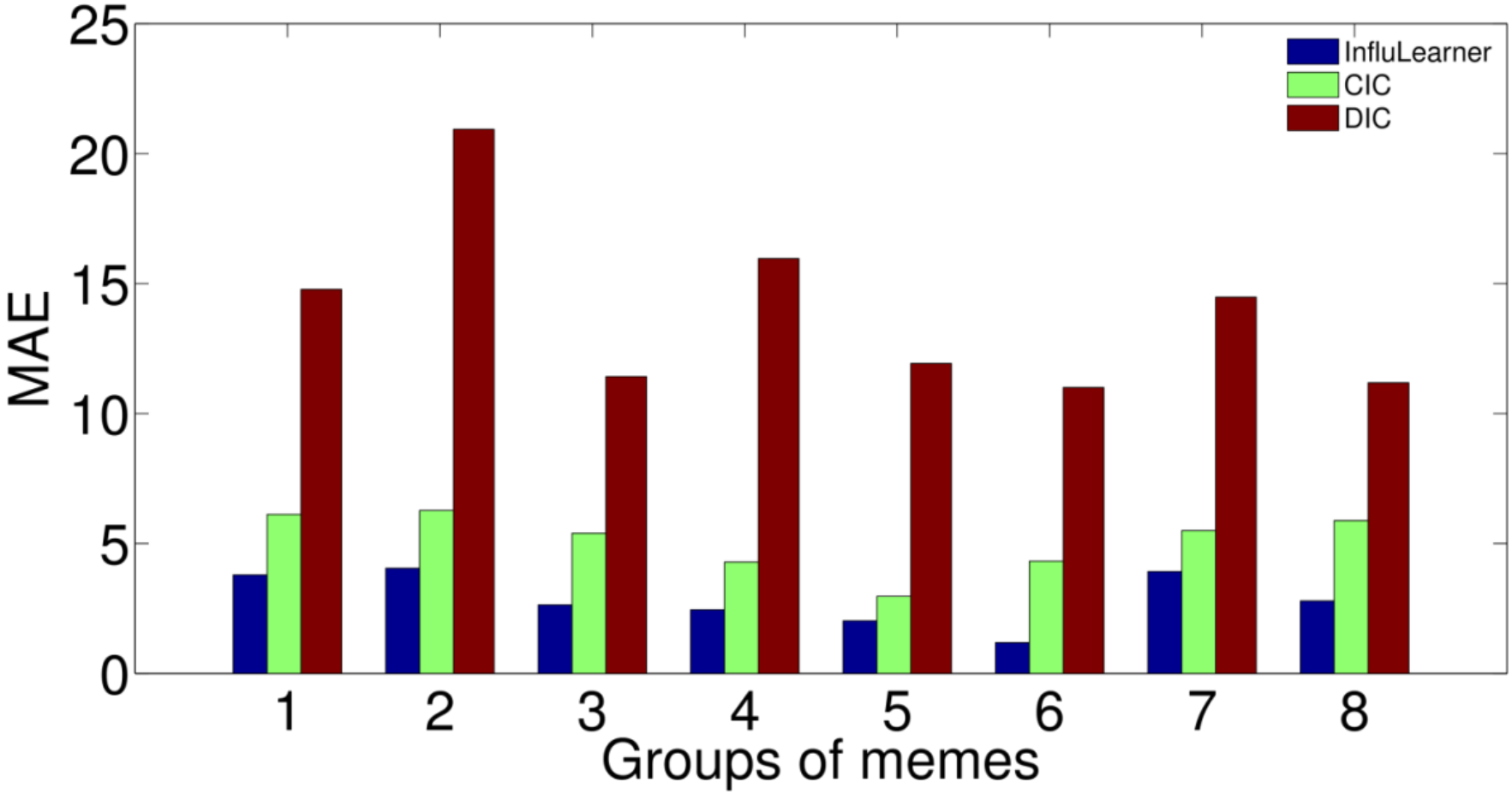
- choose random binary vectors $v_1, v_2, \dots, v_K$ from q.
- Parametrize $\hat{f}_j(S)$ as $\sum_i w_i \cdot I[\langle I_S, v_i \rangle \geq 1]$ $(\sum_i w_i \leq 1,\ w_i \geq 0)$

Learn weights via maximum conditional likelihood.

Influence estimation in real data

[Du, Liang, Balcan, Song, ICML 2014 , NIPS'14]

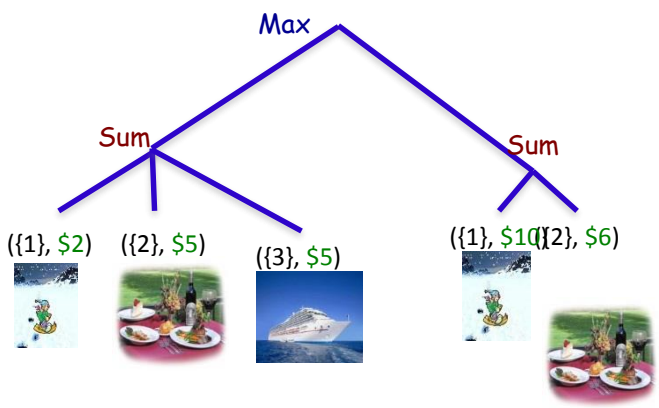
- Memetracker Dataset, blog data cascades : "apple and jobs", "tsunami earthquake", "william kate marriage"

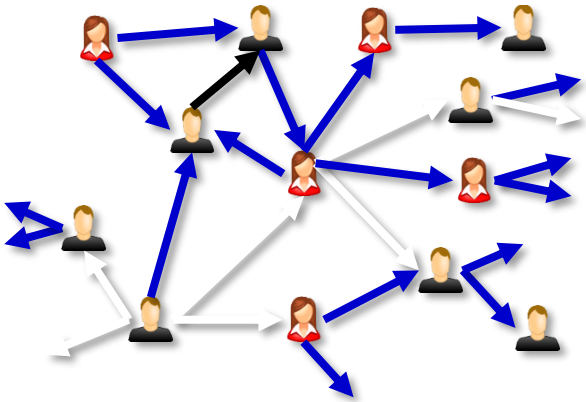


Conclusions

Learnability of submodular, other combinatorial fns

- Can model problems of interest to multi-agent systems.
- Very strong lower bounds in the worst case.
- Much better learnability results for classes with additional structure.





Conclusions

Learnability of submodular functions

- Very strong lower bounds in the worst case.
- Highlight the importance of considering the complexity of the target function.

Open Questions:

- Exploit complexity of target for better approx guarantees. [for learning and optimization]
  - What is a natural description language for submodular fns?
- Other interesting applications.