Matroids with Connections to Game Theory
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Definitions

- Matroids are combinatorial structures that generalize the notion of linear independence in matrices. Formally,
- *A* pair $\mathbf{M} := (\mathrm{E}, \mathcal{I})$ is a matroid if E is a finite set of elements and $\mathcal{I} \subseteq 2^{\mathrm{E}}$ is a non-empty family such that

  (1) if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$, and

  (2) if $I, J \in \mathcal{I}$ and $|J| < |I|$, then there exists an $i \in I \setminus J$ such that $J + i \in \mathcal{I}$.
- The subsets of E in $\mathcal{I}$ are called *independent*; those not in $\mathcal{I}$ are called *dependent*.
- Axiom 2 implies that all maximal independent sets have the same size.

  (A set B is maximally independent if $B \in \mathcal{I}$, but $B + i \notin \mathcal{I}$, for all $i \notin B$.)
- A maximal independent set is called a base of $\mathbf{M}$. All bases have the same size, which is called the rank of the matroid and denoted $\mathrm{rk}(\mathbf{M})$.
- The function $\mathrm{rank}_{\mathbf{M}} : 2^{\mathrm{E}} \to \mathbb{N}_+$ be defined by

$$\mathrm{rank}_{\mathbf{M}}(S) = \max\{|I| : I \subseteq S, I \in \mathcal{I}\}.$$

Examples

- **The uniform matroid**
  - Let $\mathrm{E} = [n]$. Let
    $$\mathcal{I} \ = \ \{I \subseteq \mathrm{E} : |I| \leq k\}.$$
    Then $\mathbf{M} = (\mathrm{E}, \mathcal{I})$ is a matroid and it is denote as $U_{k,n}$.
  - A base is any set of cardinality k (unless $k > |\mathrm{E}|$ in which the only base is E.)
  - The free matroid is one in which all sets are independent, i.e., $U_{n,n}$.
- **Linear matroids**
  - Let A be a matrix. Let E denote the index set of the columns of A. For any subset I of E, let A_I denote the submatrix of A consisting only of those columns indexed by I. Let
    $$\mathcal{I} \ = \ \{I \subseteq \mathrm{E} : rank(A_I) = |I|\},$$
    i.e. a set I is independent if the corresponding columns are independent. Then $\mathbf{M} = (\mathrm{E}, \mathcal{I})$ is a linear matroid.
  - Axiom 2 corresponds to a fundamental linear algebra property. If A_I has full rank, then its columns span a space of dimension $|I|$, and similarly for $|J|$. So, if $|J| > |I|$, there must exist a column in A_J that is not in the span of the columns of A_I; adding this column to A_I increases the rank by 1.
  - A base B corresponds to a linearly independent set of columns of cardinality $rank(A)$.

Examples

- **The graphic matroid**
  - Given a graph $G = (V, \mathrm{E})$, we define independent sets to be those subsets of edges which are forests, i.e. do not contain any cycles. $\mathbf{M} = (\mathrm{E}, \mathcal{I})$ is called the graphic matroid.
  - To check Axiom 2, notice that if F is a forest then the number of connected components of the graph (V, F) is given by $K(V, F) = |V| - |F|$. Therefore, if X and Y are 2 forests and $|Y| > |X|$ then $K(V, Y) < K(V, X)$ and therefore there must exist an edge of $Y \setminus X$ which connects two different connected components of X; adding this edge to X results in a larger forest.
  - If G is connected, then any base will correspond to a spanning tree T of the graph. If G is disconnected then a base corresponds to taking a spanning tree in each connected component of G.
- **Partition matroids**
  - Let $\mathrm{E} = [n]$ be a ground set. Assume that sets $A_1, \ldots, A_k \subseteq \mathrm{E}$ form a partition of E and let $u_1, \ldots u_k \in N$. Let

    $$\mathcal{I} \ = \ \{I \subseteq \mathrm{E} : |I \cap A_j| \leq u_j \ \forall j \in [k]\}.$$

    Then $\mathbf{M} = (\mathrm{E}, \mathcal{I})$ is a partition matroid.
  - Axiom 2 is satisfied since if $I, J \in \mathcal{I}$ and $|J| > |I|$, there must exist i such that $|J \cap A_i| \geq |I \cap A_i|$, and so adding any element e in $A_i \cap (J \setminus I)$ to I will maintain independence.
- If set $A_1, \ldots, A_k \subseteq \mathrm{E}$ are not disjoint, this is not the independent sets of a matroid. Consider taking $n = 5$, $A_1 = \{1, 2, 3\}$, $A_2 = \{3, 4, 5\}$ and $u_1 = u_2 = 2$. Then both $\{1, 2, 4, 5\}$ and $\{2, 3, 4\}$ are maximal sets in $\mathcal{I}$ but do not have the same cardinality.

Matroid Optimization

- Given a matroid $\mathrm{M} = (\mathrm{E}, \mathcal{I})$ and a cost function $c : E \rightarrow R$, we are interested in finding an independent set S of M of maximum total cost $c(S) = \sum_{e \in S} c(e)$.
- If all $c(e) \geq 0$, the problem is equivalent to finding a maximum cost *base* in the matroid. If $c(e) < 0$ for some element e then, because of Axiom 1, e will not be contained in any optimum solution, and thus we could eliminate such an element from the ground set.
  - Sort the elements (and renumber them) such that $c(e_1) \geq c(e_2) \geq \ldots c(e_{|E|})$.
  - $S_0 =$, $k = 0$.
  - For $j = 1$ to $|E|$
  - if $S_k + e_j \in \mathcal{I}$ then
    - $k \leftarrow k + 1$,
    - $S_k \leftarrow S_{k-1} + e_j$,
    - $s_k \leftarrow e_j$.
- **Theorem:** For any matroid $\mathrm{M} = (\mathrm{E}, \mathcal{I})$, the greedy algorithm finds, for every k, an independent set S_k of maximum cost among all independent sets of size k.
- **Proof:**
  - Suppose not. Let $S_k = \{s_1, s_2, \ldots, s_k\}$ with $c(s_1) \geq c(s_2) \geq \ldots \geq c(s_k)$, and suppose T_k has greater cost, i.e., $c(T_k) > c(S_k)$ where $T_k = \{t_1, t_2, \ldots, t_k\}$ with $c(t_1) \geq c(t_2) \geq \ldots \geq c(t_k)$.
  - Let p be the first index such that $c(t_p) > c(s_p)$. Let $A = \{t_1, t_2, \ldots, t_p\}$ and $B = \{s_1, s_2, \ldots, s_{p-1}\}$. Since $|A| > |B|$, there exists $t_i \in B$ such that $B + t_i \in \mathcal{I}$. Since $c(t_i) \geq c(t_p) > c(s_p)$, t_i should have been selected when it was considered.

    Formally, when t_i was considered, the greedy algorithm checked whether t_i could be added to the current set at the time, say S. But since $S \subseteq B$, adding t_i to S should have resulted in an independent set since its addition to B results in an independent set. This gives the contradiction.
- For graphic matroids, the greedy algorithm corresponds to Kruskal's alg.

The Rank Function of a Matroid

- The function $\mathrm{rank}_{\mathbf{M}} : 2^{\mathrm{E}} \to \mathbb{N}_+$ defined by

$$\mathrm{rank}_{\mathbf{M}}(S) = \max\{|I| : I \subseteq S, I \in \mathcal{I}\}.$$

  is called the rank function of $\mathbf{M} = (\mathrm{E}, \mathcal{I})$. For simplicity, denote $\mathrm{rank}_{\mathbf{M}}(S)$ by $r(S)$.

- **Examples:**
  1. For linear matroids, the rank of S is precisely the rank of matrix A_S corresponding to the columns A in S.
  2. For partition matroids, $r(S) = \sum_{i=1}^{k} \min(|E_i \cap X|, u_i)$.
  3. For the graphic matroids defined on $G = (V, E)$, the rank function is $\mathrm{rank}(F) = n - K(V; F)$, where $K(V; F)$ denotes the number of connected components (including isolated vertices) of the graph with edges F.

- **Theorem:** The rank function of a matroid is submodular.

- **Proof:**
  - Clearly $r(\emptyset) = 0$, r is non-decreasing, $r(S \cup \{j\}) - r(S) \leq 1$.
  - It suffices to show $r(S \cup \{j\}) - r(S) \geq r(S \cup \{j, k\}) - r(S \cup \{k\})$.
    - The inequality holds when $r(S \cup \{j\}) - r(S) = 1$.
    - Assume $r(S \cup \{j\}) = r(S) = p$ and $r(S \cup \{j, k\}) - r(S \cup \{k\}) = 1$.
      We have either $r(S \cup \{j, k\}) = p + 2, p + 1$.
      1. If $r(S \cup \{j, k\}) = p + 2$, then $r(S \cup \{j, k\}) - r(S \cup \{j\}) = 2$, contradiction.
      2. Assume $r(S \cup \{j, k\}) = p + 1$ and $r(S \cup \{k\}) = p$.
         Let B be a basis for S. Since $r(S \cup \{k\}) = r(S \cup \{j\}) = p$ it follows that $B \cup \{j\}$ and $B \cup \{k\}$ are dependent sets. Thus B is a basis for $B \cup \{j, k\}$, so $r(S \cup \{j, k\}) = p$, contradiction.

Best response dynamics in matroid congestion games

- Matroid congestion games: congestion games such that the set of strategies of each player consists of the bases of a matroid over the set of resources (*any kind* of delay functions).
- Examples:
  - Set cover games are matroid congestion games (think about uniform matroids).
  - Symmetric fair cost-sharing games might *not* be matroid congestion games (a player could have two different allowable strategies (two paths) of different cardinality; this cannot happen in a matroid congestion game since any two bases of a matroid have the same cardinality).
- **Theorem:** In a matroid congestion game the players reach a Nash after at most $n^2 m \max_i \mathrm{rank}(\mathbf{M}_i)$ best responses.
- Informally speaking, one can show that the matroid property is also to necessary guarantee fast convergence.
- See the paper: *On the Impact of Combinatorial Structure on Congestion Games*. by H. Ackermann, H. Röglin, and B. Vöcking, Journal of the ACM, 2008.