# Nash Equilibria in General Sum Games

In this lecture we study basic solution concepts in game theory, namely the notion of dominant strategy and Nash equilibrium. We consider rational players in a competitive environment. A rational player is a player with priorities (or utility) that tries to maximize the utility (or minimize cost) while considering that other players are also rational. A competitive environment is an environment with multiple rational players.

## An Example: The Prisoner's Dilemma

There are two prisoners that committed a crime. If they both do not confess, they get a low punishment. If they both confess, they get a more severe punishment. If one confesses and the other does not, then the one that confesses gets a low punishment and the other gets a very severe punishment. The game can be formalized in the following matrix, each entry includes a pair, the first is the cost to the first player and the second is the cost to the second player.

Table 1: Cost Matrix for Prisoner Dilemma

| | Confess | Silent |
|---|---|---|
| Confess | $(4, 4)$ | $(1, 5)$ |
| Silent | $(5, 1)$ | $(2, 2)$ |

Game theory predicts the case where both prisoners confess (4,4): player $i$ doesn't know what the other player chooses. Should the other player confess, then player $i$ can either confess (4 years imprisonment) or not confess (5 years). Should the other player choose to remain silent, then player $i$ can confess (1 year) or keep silent (2 years). Thus, in both cases it is better to *confess*. This is an example of a strong dominant strategy.

# Strategic Games

A strategic game is a model for decision making where there are $N$ players, each choosing an action. A player's action is chosen once and cannot be changed afterwards.

Each player $i$ can choose an action $a_i$ from a set of actions $A_i$. Let $A$ be the set of all possible action vectors $\times_{j \in N} A_j$. Each player has either a utility function $u_i : A \to \mathbb{R}$ which is to be maximized or alternatively, a cost function $c_i : A \to \mathbb{R}$ which should be minimized.

**Model** A **Strategic Game** is a triplet $\langle N, (A_i)_{i=1}^N, (u_i)_{i=1}^N \rangle$ where $N = \{1...n\}$ is the set of $n$ players, $A_i$ is the finite set of actions for player $i$, and $u_i$ is the utility function of player $i$.

Several player behaviors are assumed in a strategic game:

- The game is played only once.
- Each player "knows" the game (each player knows all the actions and the possible outcomes of the game).
- The players are rational. A rational player is a player that plays selfishly, wanting to maximize her own benefit of the game (the utility function).
- All the players choose their actions simultaneously (but do not know the other players current choices).

We now define a dominating action. An action is dominating if it is better than any other action, regardless of the action of the other players.

**Definition 1** *Action $a_i$ is a* **Weak Dominant Strategy** *for player $i$ if*

$$\forall b_{-i} \in A_{-i}. \forall b_i \in A_i : u_i(b_{-i}, b_i) \leq u_i(b_{-i}, a_i)$$

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Where $(a_{-i}) = (a_1, a_2, ..., a_{i-1}, a_{i+1}, ..., a_n)$.

However, not all games have dominant strategies.

## An Example: Battle of the Sexes

In this game, two players (of different gender) need to coordinate on an event (sports or opera). They both prefer to go to the same event together(gaining a value of 2 each if they go to the same event, or 0 if not), but they have a different preference between the events (value 2 for preferred event and 1 for the other).

This game does not have a dominating action for any of the player. It has two pure Nash Equilibria points: (Sports, Sports) and (Opera, Opera).

Table 2: Utility Matrix for the Battle of the Sexes Game

| | Sports | Opera |
|---|---|---|
| Sports | $(4, 3)$ | $(2, 2)$ |
| Opera | $(1, 1)$ | $(3, 4)$ |

## Nash Equilibria

**Definition 2** *A pure Nash equilibrium is a joint action $a \in A$ such that:*

1. $\forall i \in N, \forall b_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i})$ *or alternatively,*
2. $\forall i \in N : a_i \in BR(a_{-i})$.

*Namely, no player can unilateraly improve his payoff.*

However, not all games have a pure Nash equilibrium.

## An Example: Matching Pennies

In this game each player select Head or Tails. The row player wins if they match, and the column player wins if they mismatch (Matching Pennies).

Table 3: Utility Matrix for the Matching Pennies Game

| | Head | Tail |
|---|---|---|
| Head | $(1, -1)$ | $(-1, 1)$ |
| Tail | $(-1, 1)$ | $(1, -1)$ |

It is easy to verify that there is no pure Nash Equilibrium point in this game. Also, this is a zero sum game (the sum of the profits of the player for each possible outcome is 0).

If we allow each player to randomly choose an outcome with a pre-defined probability the game will be an example of a mixed strategy game. Let us assume that player 1 chooses "Head" with probability $p$ and "Tail" with probability $1 - p$ and that player 2 chooses "Head" with probability $q$ and "Tail" with probability $1 - q$. In this case each player wants to maximize its expected utility.

Assume that player 2 plays "Head" with probability $q$ and "Tail" with probability $1 - q$. If player 1 plays "Head" the outcome will be $-1$ with probability $q$ and 1 with probability $1 - q$ thus the expected utility is $2q - 1$. If player 1 plays "Tail" the outcome will be $-1$ with probability $1 - q$ and 1 with probability $q$ thus the expected utility is $1 - 2q$. So if $q = 1/2$, these two quantities are both equal to 0. So, if the second player plays with $q = 1/2$, the first player has no incentive to deviate from $(1/2, 1/2)$ (in fact he has no incentive to deviate from any strategy).

Similarly, assume that player 1 plays "Head" with probability $p$ and "Tail" with probability $1-p$. If player 2 plays "Head" his the expected utility is $1-2p$ and if player 2 plays "Head" his the expected utility $2p-1$. So if $p = 1/2$, these two quantities are both equal to 0. So, if the first player plays with $p = 1/2$, the second player has no incentive to deviate from $(1/2, 1/2)$ (in fact he has no incentive to deviate from any strategy).

So, $(1/2, 1/2), (1/2, 1/2)$ is a mixed Nash equilibrium. This is in fact the only Nash equilibrium of the game. Assume by contradiction that $q > 1/2$. Note that if $p \neq 1$ we cannot be at an equilibrium since the first player would rather play heads to what he is doing now. On the other hand if $p = 1$, the column payer would prefer $q = 0$ , contradiction.

## Mixed Nash Equilibria

Recall that a finite strategic game consists of the following:

- A finite set of players, namely $N = \{1, \ldots, n\}$.
- For every player $i$, a set of actions $A_i = \{a_{i1}, \ldots, a_{im}\}$.
- The set $A = \otimes_{i=1}^n A_i$ of joint actions.
- For every player $i$, a utility function $u_i : A \to \Re$.

A mixed strategy for player $i$ is a random variable over his actions. The set of such strategies is denoted $\triangle(A_i)$. Letting every player have his own mixed strategy (independent of the others) leads to the set of joint mixed strategies, denoted $\triangle(A) = \otimes_{i=1}^n \triangle(A_i)$.

Every joint mixed strategy $p \in \triangle(A)$ consists of $n$ vectors $\vec{p_1}, \ldots, \vec{p_n}$, where $\vec{p_i}$ defines the distribution played by player $i$. Taking the expectation over the given distribution, we define the utility for player $i$ by

$$u_i(p) = E_{a \sim p}\left[u_i(a)\right] = \sum_{a \in A} p(a) u_i(a) = \sum_{a \in A} \left( \prod_{i=1}^{n} \vec{p_i}(a_i) \right) u_i(a)$$

We can now define a Nash Equilibrium (NE) as a joint strategy where no player profits from unilaterally changing his strategy:

**Definition 3** *A joint mixed strategy $p^* \in \triangle(A)$ is NE, if for every player $1 \leq i \leq n$ it holds that*

$$\forall q_i \in \triangle(A_i) \qquad u_i(p^*) \geq u_i(p^*_{-i}, q_i)$$

*or equivalently*

$$\forall a_i \in A_i \qquad u_i(p^*) \geq u_i(p^*_{-i}, a_i)$$

## Existence Theorem

**Theorem 1** *Every finite game has a (mixed-strategy) Nash Equilibrium.*

This section shall outline a proof of this theorem. We begin with a statement of Brouwer's Lemma and conclude with the proof.

**Lemma 1** *(Brouwer) Let $f : B \to B$ be a continuous function from a non-empty, compact, convex set $B \subset \Re^n$ to itself. Then there is $x^* \in S$ such that $x^* = f(x^*)$ (i.e. $x^*$ is a fixed point of $f$).*

To demonstrate that the conditions are necessary, we show a few examples:

**When** $B$ is not bounded: Consider $f(x) = x + 1$ for $x \in \Re$. Then, there is obviously no fixed point.

**When** $B$ is not closed: Consider $f(x) = x/2$ for $x \in (0, 1]$. Then, although $x = 0$ is a fixed point, it is not in the domain.

**When** $B$ is not convex: Consider a circle in 2D with a hole in its center (i.e. a ring). Let $f$ rotate the ring by some angle. Then, there is obviously no fixed point.

## Proof of Existence of Nash Equilibrium

For $1 \le i \le n$, $j \in A_i$, $p \in \triangle(A)$ we define

$$g_{ij}(p) = \max\{u_i(p_{-i}, a_{ij}) - u_i(p), 0\}$$

to be the gain for player $i$ from switching to the deterministic action $a_{ij}$, when $p$ is the joint strategy (if this switch is indeed profitable). We can now define a continuous map between mixed strategies $y : \triangle(A) \to \triangle(A)$ by

$$y_{ij}(p) = \frac{p_{ij} + g_{ij}(p)}{1 + \sum_{j'=1}^{m} g_{ij'}(p)}.$$

Observe that:

- For every player $i$ and action $a_{ij}$, the mapping $y_{ij}(p)$ is continuous (w.r.t. $p$). This is due to the fact that $u_i(p)$ is obviously continuous, making $g_{ij}(p)$ and consequently $y_{ij}(p)$ continuous.
- For every player $i$, the vector $(y_{ij}(p))_{j=1}^{m}$ is a distribution, i.e. it is in $\triangle(A_i)$. This is due to the fact that the denominator of $y_{ij}(p)$ is a normalizing constant for any given $i$.

Therefore $y$ fulfills the conditions of Brouwer's Lemma. Using the lemma, we conclude that there is a fixed point $p$ for $y$. This point satisfies

$$p_{ij} = \frac{p_{ij} + g_{ij}(p)}{1 + \sum_{j'=1}^{m} g_{ij'}(p)}.$$

This is possible only in one of the following cases. Either $g_{ij}(p) = 0$ for every $i$ and $j$, in which case we have an equilibrium (since no one can profit from changing his strategy). Otherwise, assume there is a player $i$ s.t. $\sum_{j'=1}^{m} g_{ij'}(p) > 0$. Then,

$$p_{ij}\left(1 + \sum_{j=1}^{m} g_{ij'}(p)\right) = p_{ij} + g_{ij}(p)$$

or

$$p_{ij}\left(\sum_{j'=1}^{m} g_{ij'}(p)\right) = g_{ij}(p).$$

This means that $g_{ij}(p) = 0$ iff $p_{ij} = 0$, and therefore $p_{ij} > 0 \Rightarrow g_{ij}(p) > 0$. However, this is **impossible** by the definition of $g_{ij}(p)$: $g_{ij}(p) \neq 0 \Longrightarrow u_i(p_{-i}, a_{ij}) > u_i(p)$ for every $j$ in $p_i$'s support. Taking the mean of these inequalities we get

$$\sum_j p_{ij} u_i(p_{-i}, a_{ij}) > \sum_j p_{ij} u_i(p).$$

But both sides are equal since

$$\sum_j p_{ij} u_i(p) = u_i(p) \sum_j p_{ij} = u_i(p)$$

and by definition

$$\sum_j p_{ij} u_i(p_{-i}, a_{ij}) = u_i(p)$$

so we get the contradiction $u_i(p) < u_i(p)$. Therefore, it cannot be that player $i$ can profit from every pure action in $\vec{p_i}$'s support (with respect to the mean).

We are therefore left with the former possibility, i.e. $g_{ij}(p) = 0$ for all $i$ and $j$, implying a NE. ■

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