This homework is due by the start of class on October 28th 2010. You can either submit the homework via the course page on T-Square or hand it in at the beginning of the class on October 28th 2010. Start early!

Groundrules:

- Your work will be graded on correctness, clarity, and conciseness.
- You may collaborate with others on this problem set. However, you must write your own solutions and list your collaborators/sources for each problem.

Problems:

1. **VC-dimension of linear separators:** In this problem you will prove that the VC-dimension of the class $H_n$ of halfspaces (another term for linear threshold functions) in $n$ dimensions is $n + 1$. We will use the following definition: The convex hull of a set of points $S$ is the set of all convex combinations of points in $S$; this is the set of all points that can be written as $\sum_{i \in S} \lambda_i x_i$, where each $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$. It is not hard to see that if a halfspace has all points from a set $S$ on one side, then the entire convex hull of $S$ must be on that side as well.

   (a) **[lower bound]** Prove that $\text{VC-dim}(H_n) \geq n + 1$ by presenting a set of $n + 1$ points in $n$-dimensional space such that one can partition that set with halfspaces in all possible ways. (And, show how one can partition the set in any desired way.)

   (b) **[upper bound part 1]** The following is “Radon’s Theorem,” from the 1920’s.

   **Theorem.** Let $S$ be a set of $n + 2$ points in $n$ dimensions. Then $S$ can be partitioned into two (disjoint) subsets $S_1$ and $S_2$ whose convex hulls intersect.

   Show that Radon’s Theorem implies that the VC-dimension of halfspaces is at most $n + 1$. Conclude that $\text{VC-dim}(H_n) = n + 1$.

   (c) **[upper bound part 2]** Now we prove Radon’s Theorem. We will need the following standard fact from linear algebra. If $x_1, \ldots, x_{n+1}$ are $n + 1$ points in $n$-dimensional space, then they are linearly dependent. That is, there exist real values $\lambda_1, \ldots, \lambda_{n+1}$ not all zero such that $\lambda_1 x_1 + \ldots + \lambda_{n+1} x_{n+1} = 0$.

   You may now prove Radon’s Theorem however you wish. However, as a suggested first step, prove the following. For any set of $n + 2$ points $x_1, \ldots, x_{n+2}$ in $n$-dimensional space, there exist $\lambda_1, \ldots, \lambda_{n+2}$ not all zero such that $\sum_i \lambda_i x_i = 0$ and $\sum_i \lambda_i = 0$. (This is called affine dependence.)

2. **Symmetric congestion games:** Consider a network congestion game defined on a directed graph $G = (V, E)$. Assume each player $i$ wants to get from source $o$ and to destination $t$ (so all players have the same set of strategies, which is the set of paths from $o$ to $t$). Assume that the cost of each edge is monotonically increasing with the number of players using that
edge. Show that these games have a pure Nash equilibrium and provide a polynomial time algorithm to compute a pure Nash equilibrium.

**Hints:** Try to compute the solution that minimizes the potential function. Think of min-cost network flow.

**Extra Credit:**

1. **Boosting and Game Theory:** Think about the boosting process in relation to the decision theoretic experts model and derive a boosting algorithm based on this connection.

   **Note:** You are welcome to read the paper “Game Theory, On-line Prediction and Boosting” by Rob Schapire and Yoav Freund, but please write the solution in your own words.