Robust lossy detection using sparse measurements

The regular case

Balakrishnan Narayanaswamy
(Joint work with Rohit Negi and Pradeep Khosla)

There's Nothing So Practical As a Good Theory
-Kurt Lewin

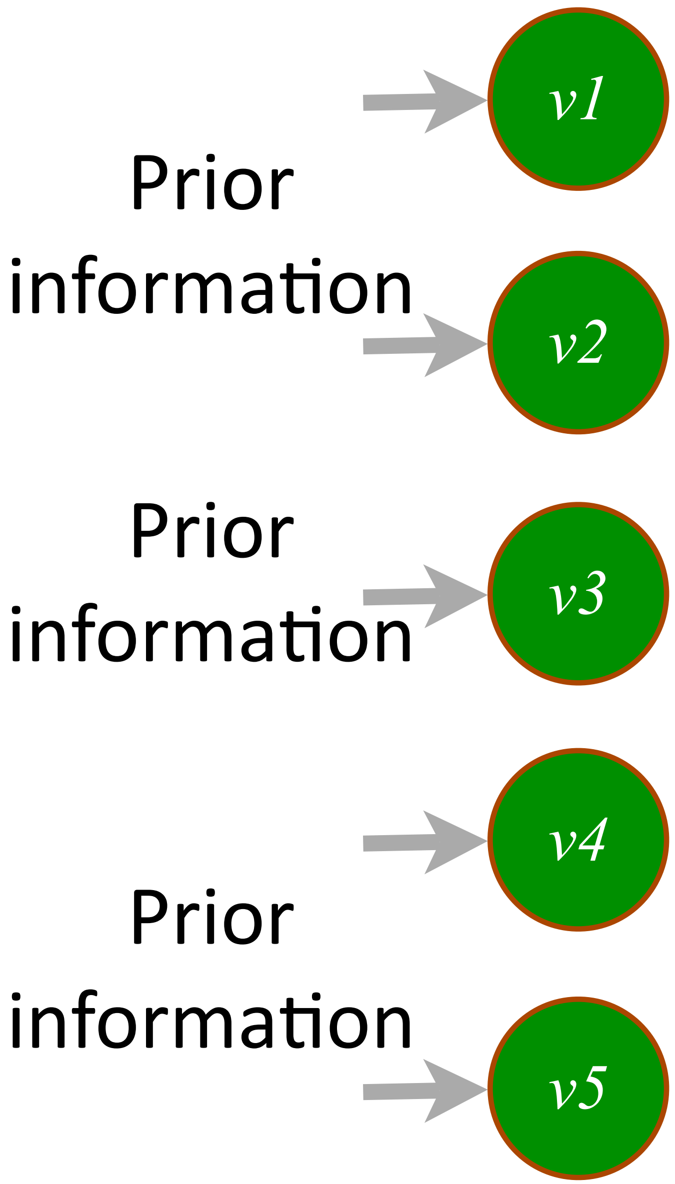
Carnegie Mellon

Electrical & Computer ENGINEERING

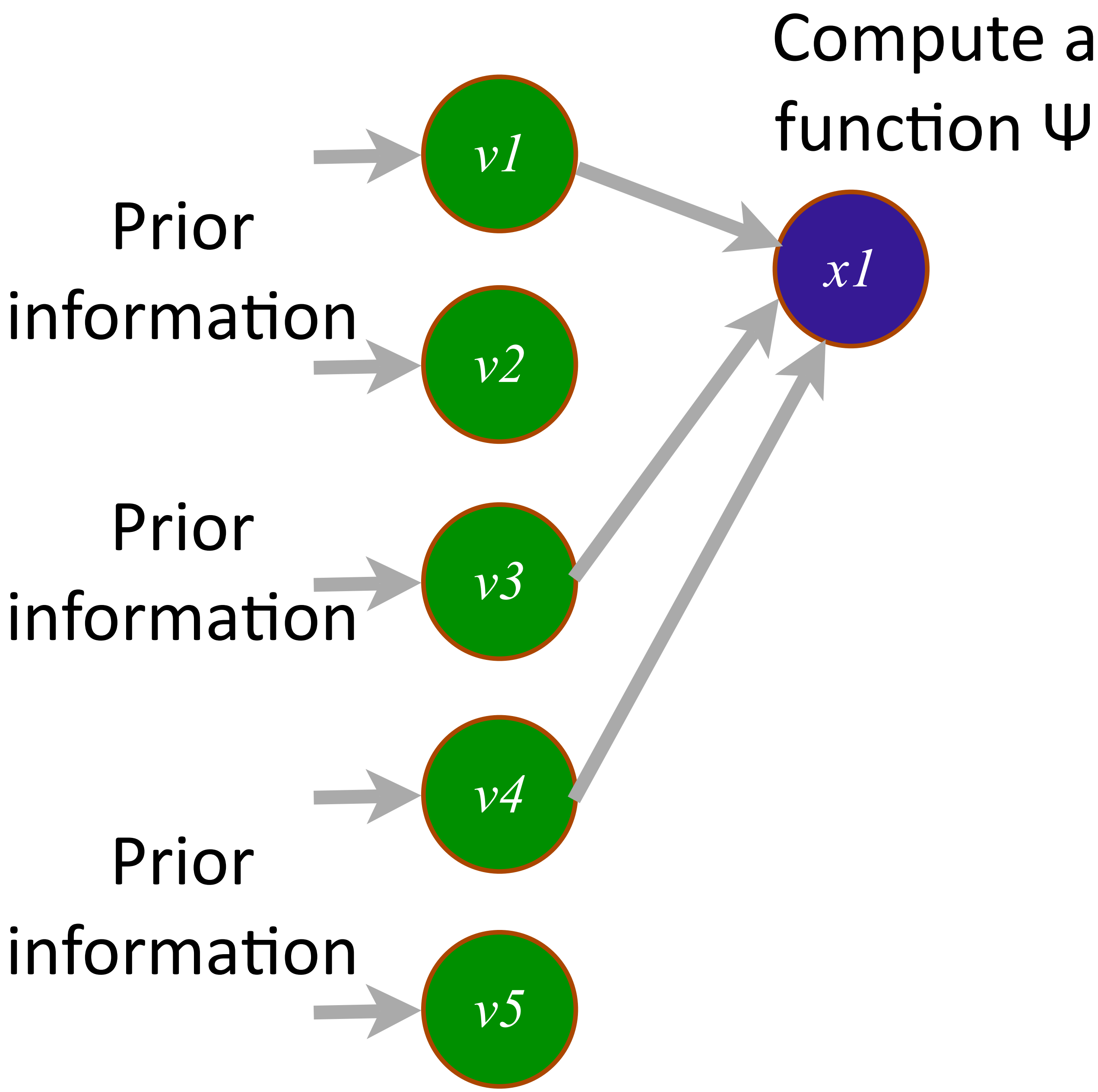
Carnegie Mellon CyLab
CONFIDENCE FOR A NETWORKED WORLD

A graphical model for measurements

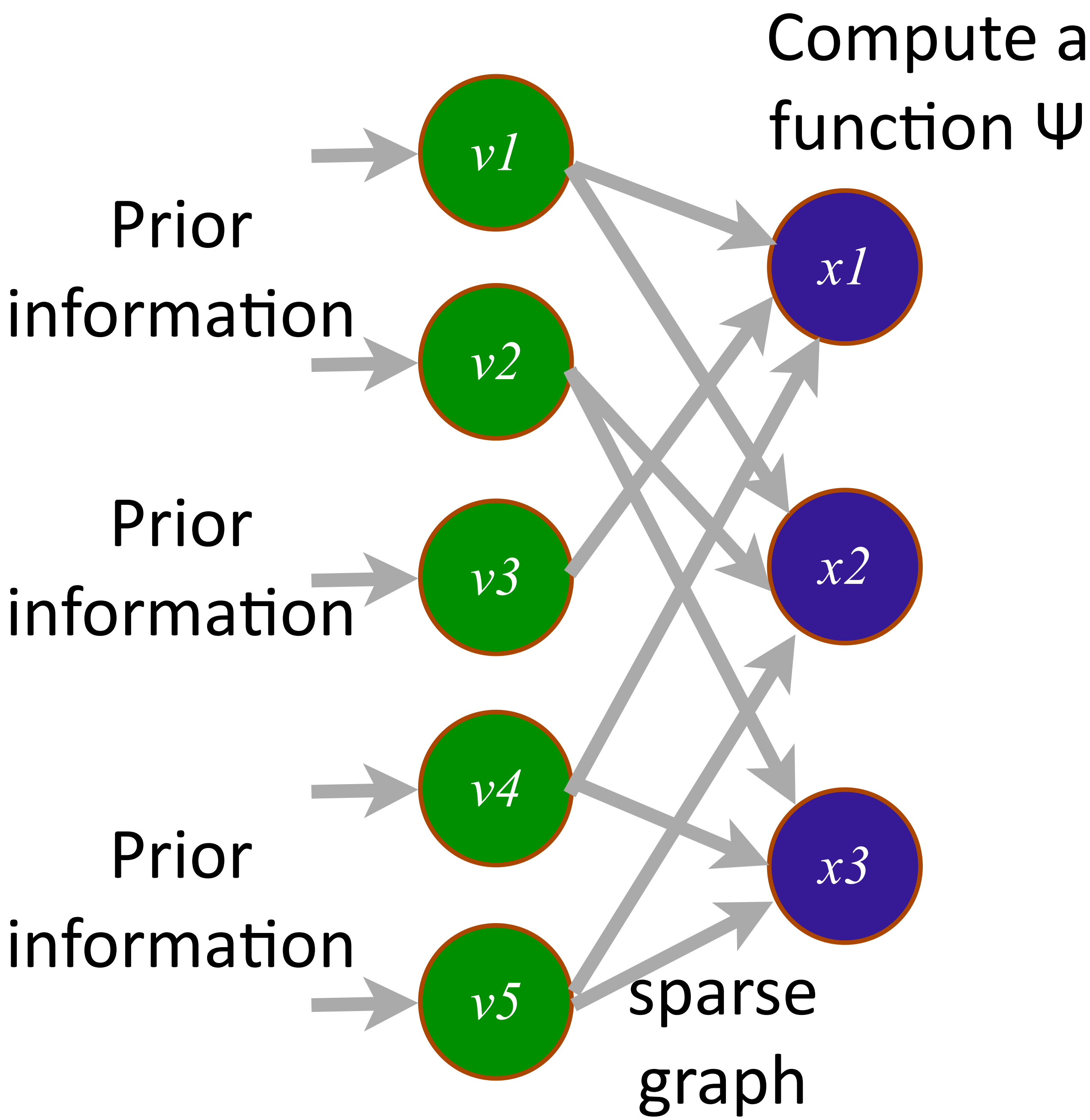
A graphical model for measurements



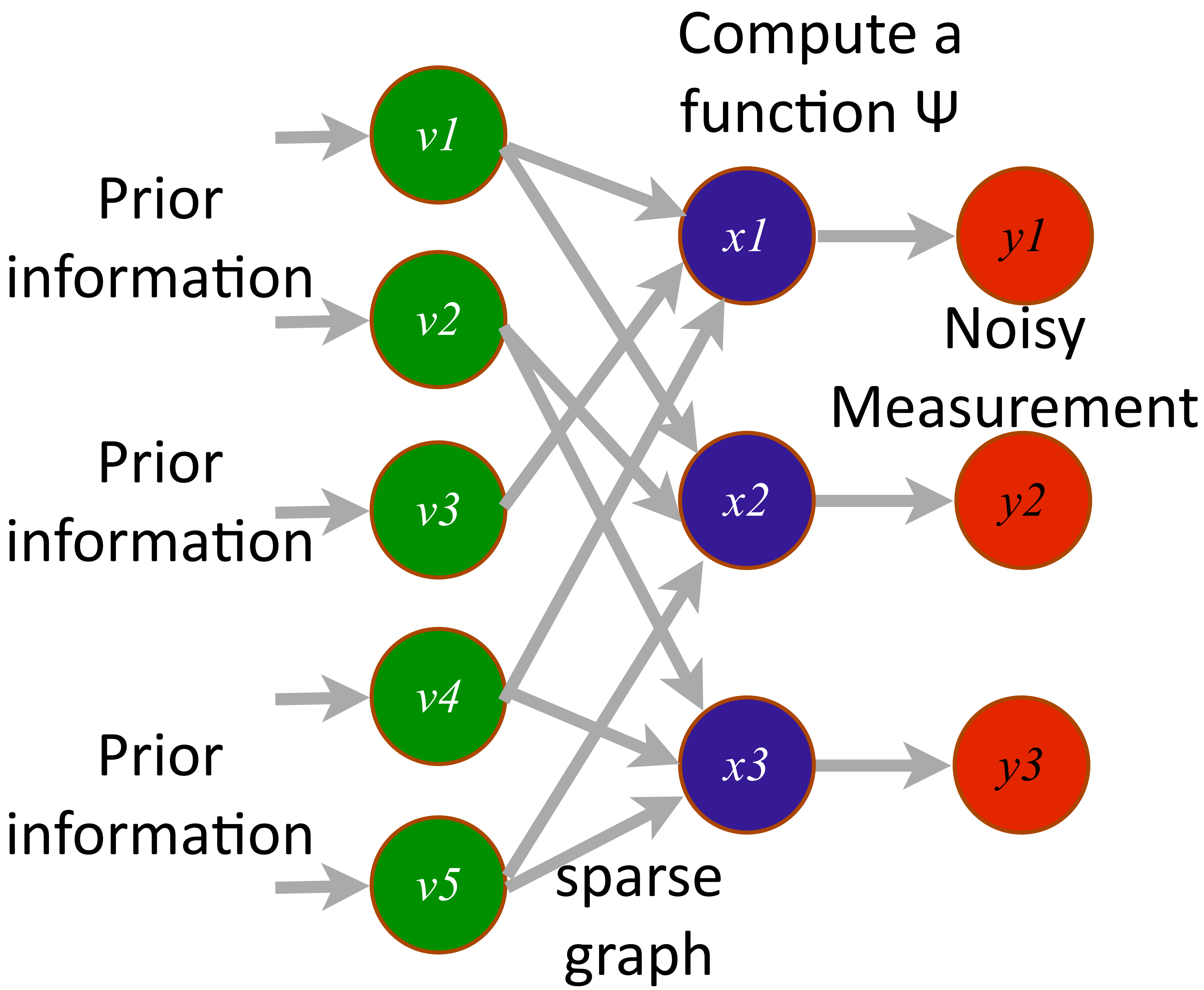
A graphical model for measurements



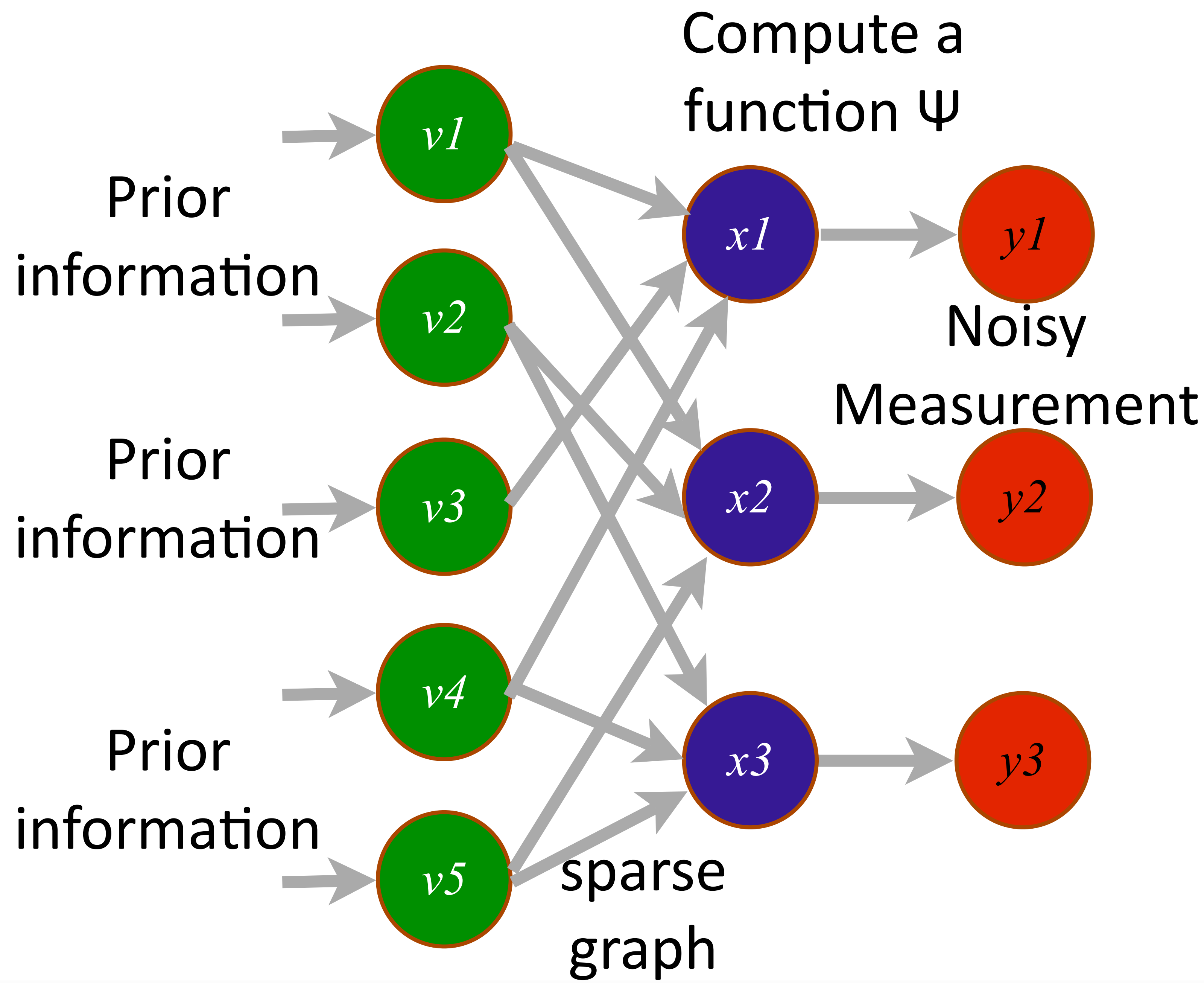
A graphical model for measurements



A graphical model for measurements



A graphical model for measurements



When can we reconstruct the input from the noisy, sparse measurements ?

The need for population wide genetic screening

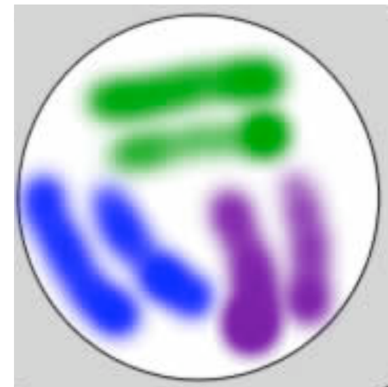
The need for population wide genetic screening

Diploid

Two Homologous copies of each chromosome

The need for population wide genetic screening

Diploid

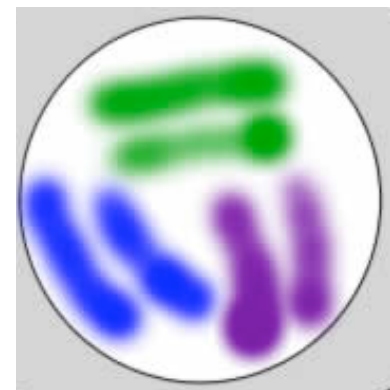


Two Homologous copies of each chromosome

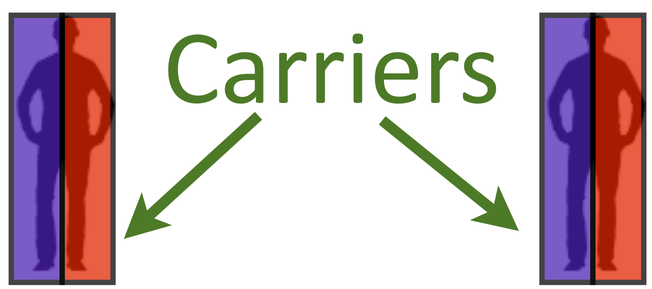
Sickle-cell anemia, Albinism, Cystic Fibrosis, Tay-Sachs Disease, Gaucher's Disease

The need for population wide genetic screening

Diploid



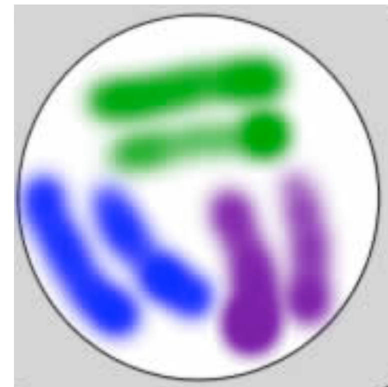
Two Homologous copies of each chromosome



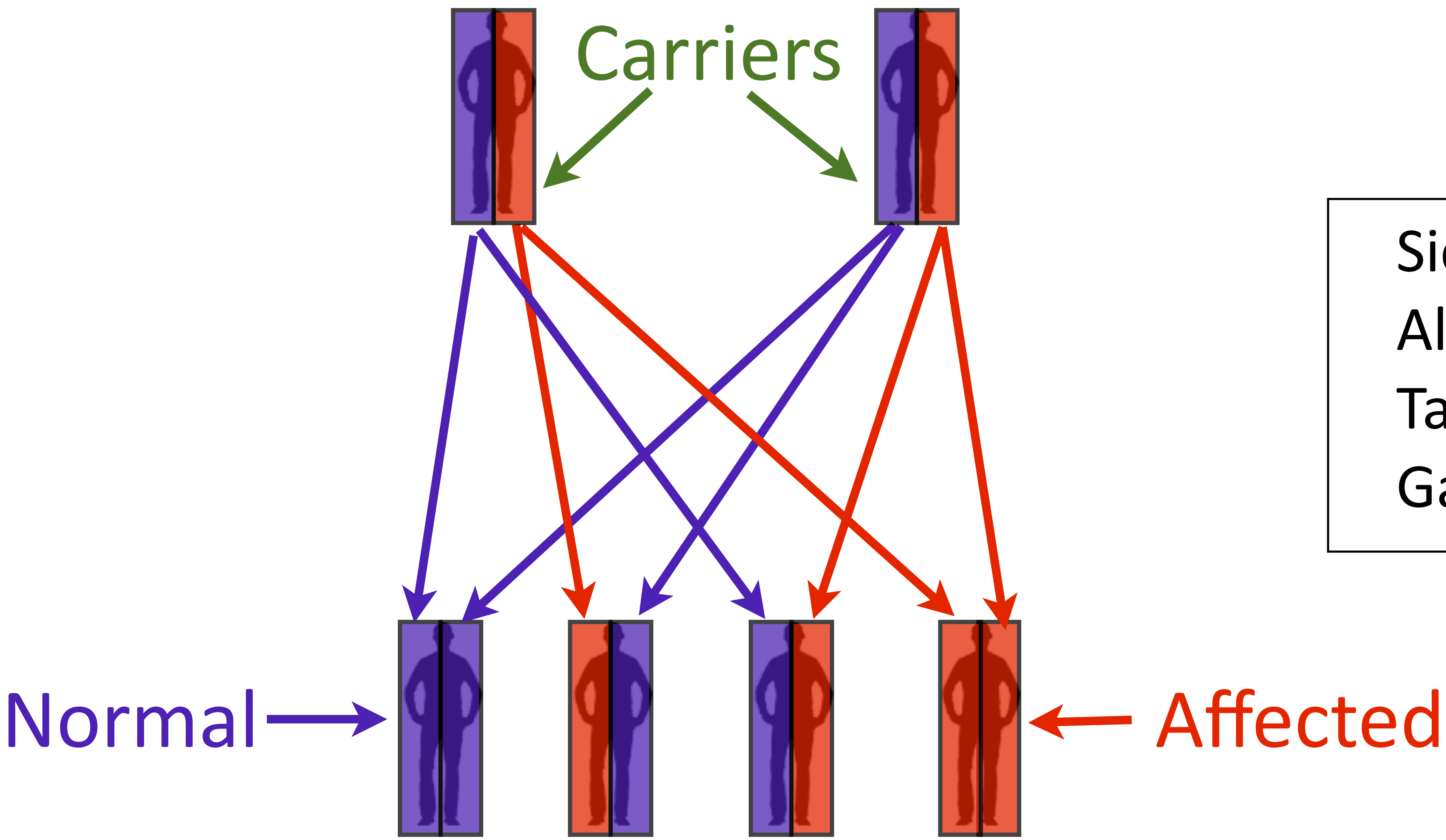
Sickle-cell anemia,
Albinism, Cystic Fibrosis,
Tay-Sachs Disease,
Gaucher's Disease

The need for population wide genetic screening

Diploid

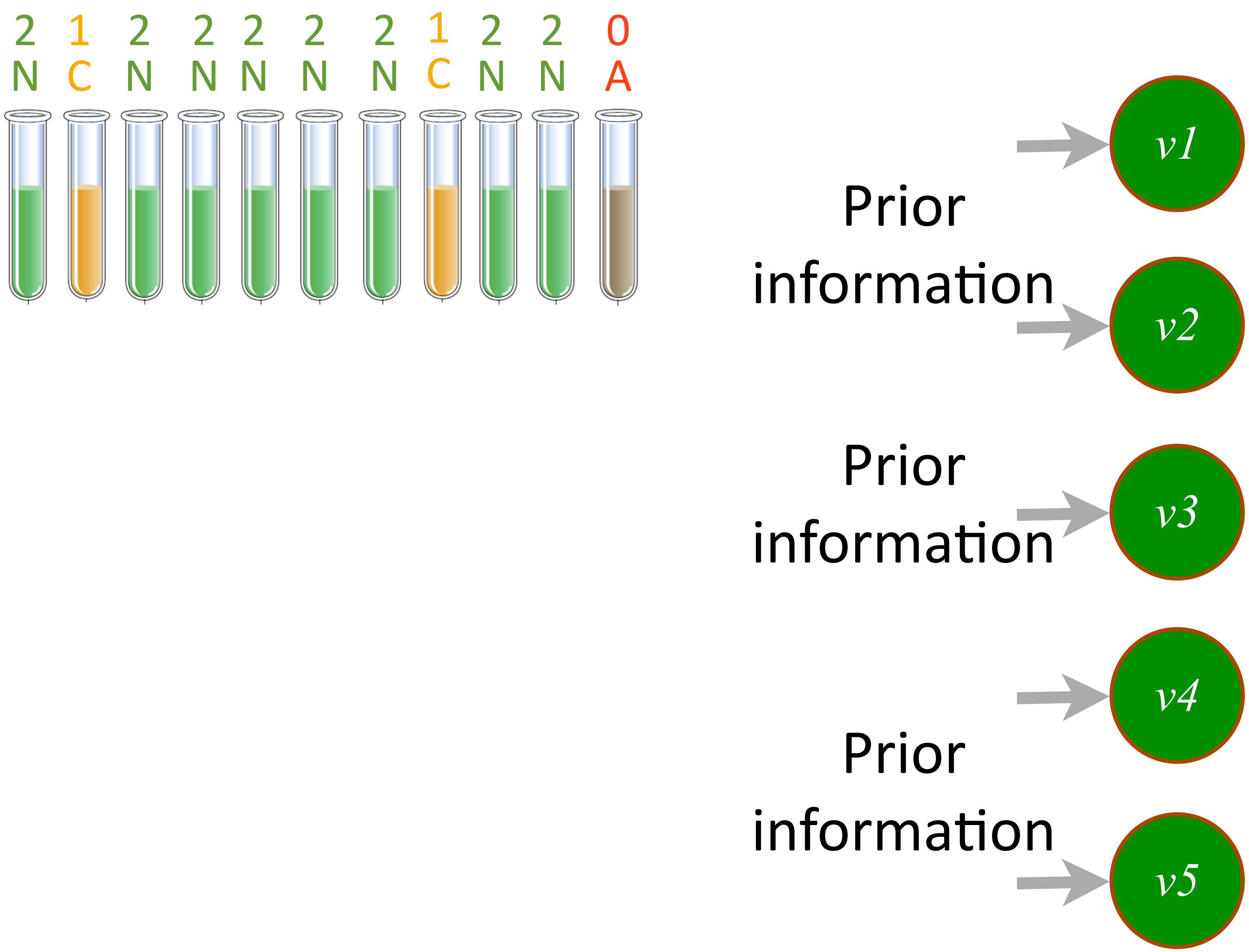


Two Homologous copies of each chromosome

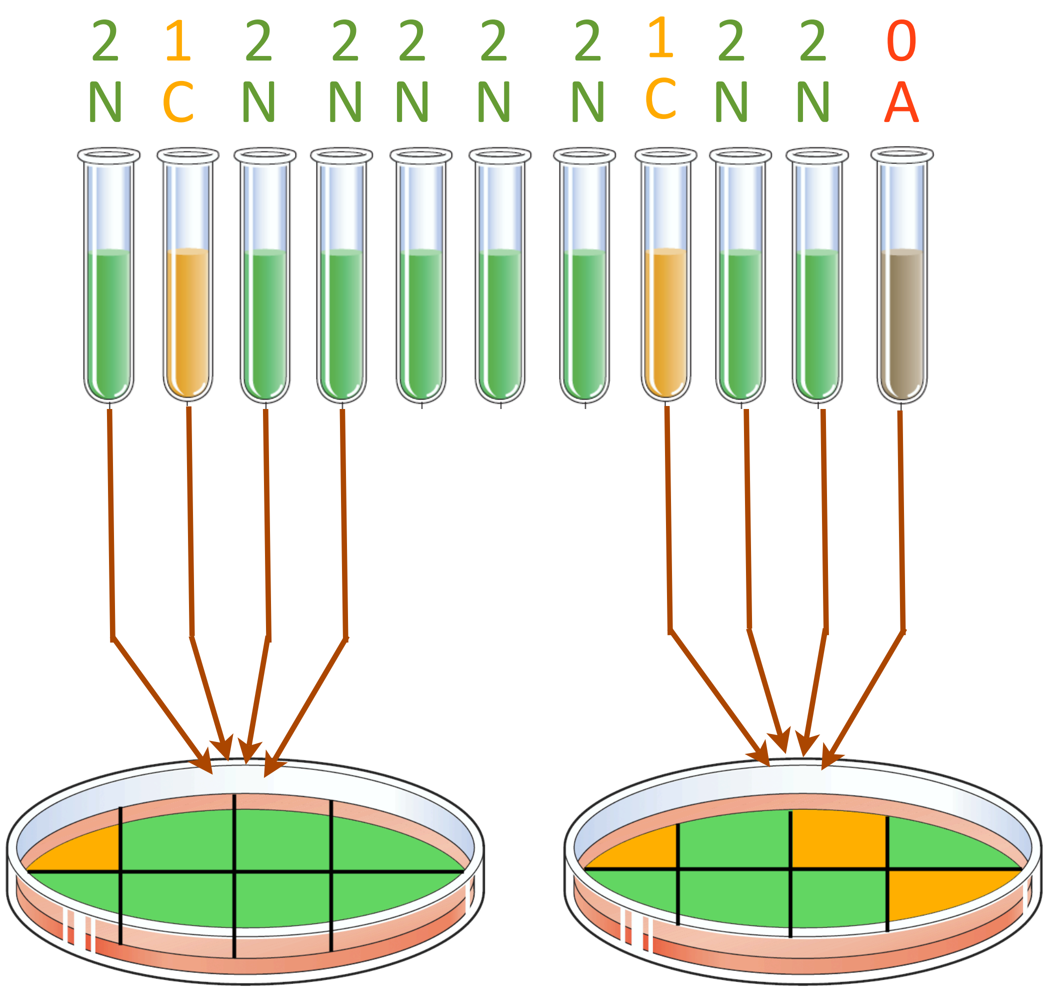


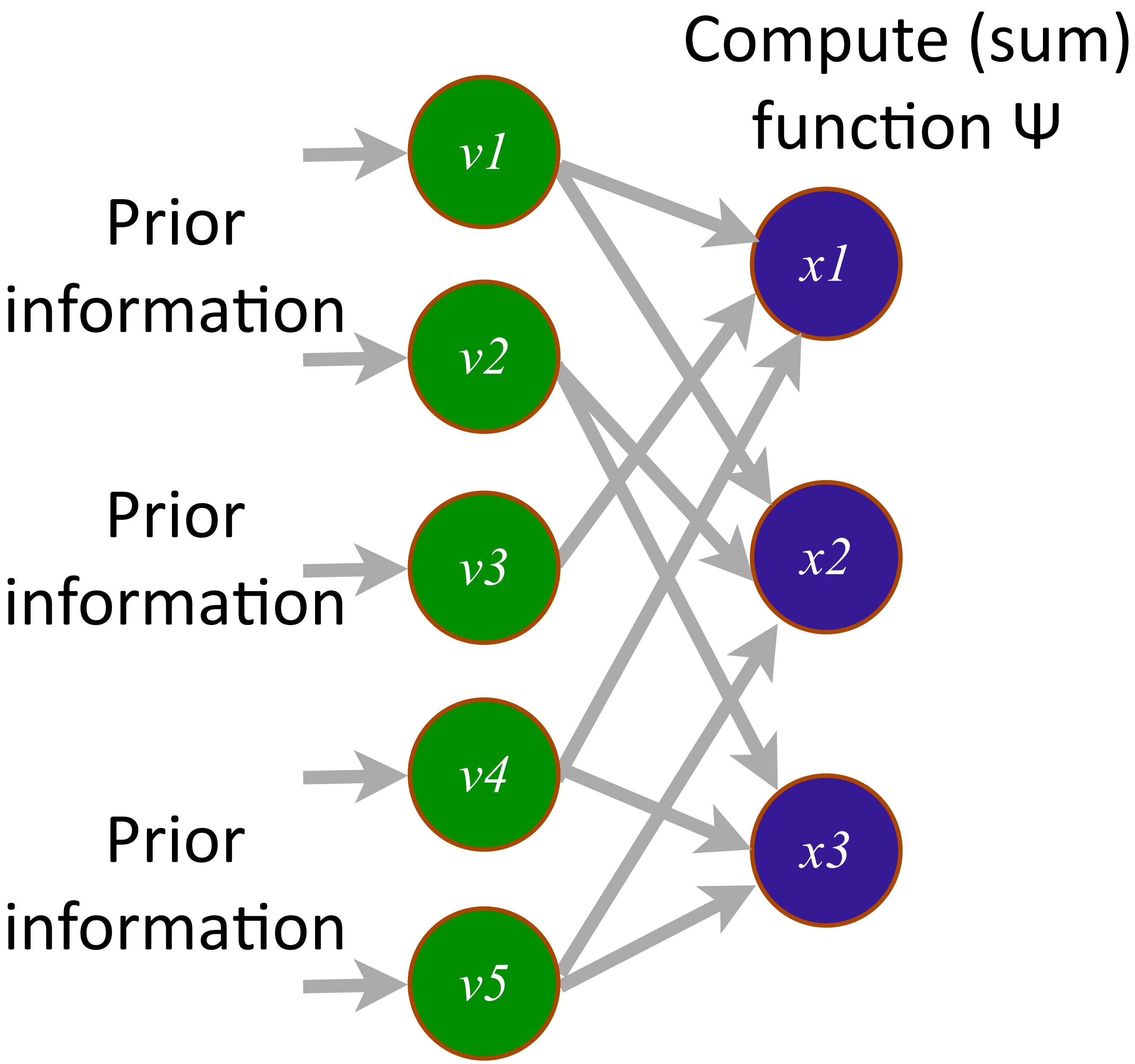
Sickle-cell anemia, Albinism, Cystic Fibrosis, Tay-Sachs Disease, Gaucher's Disease

Pooling designs as a sparse graphical model

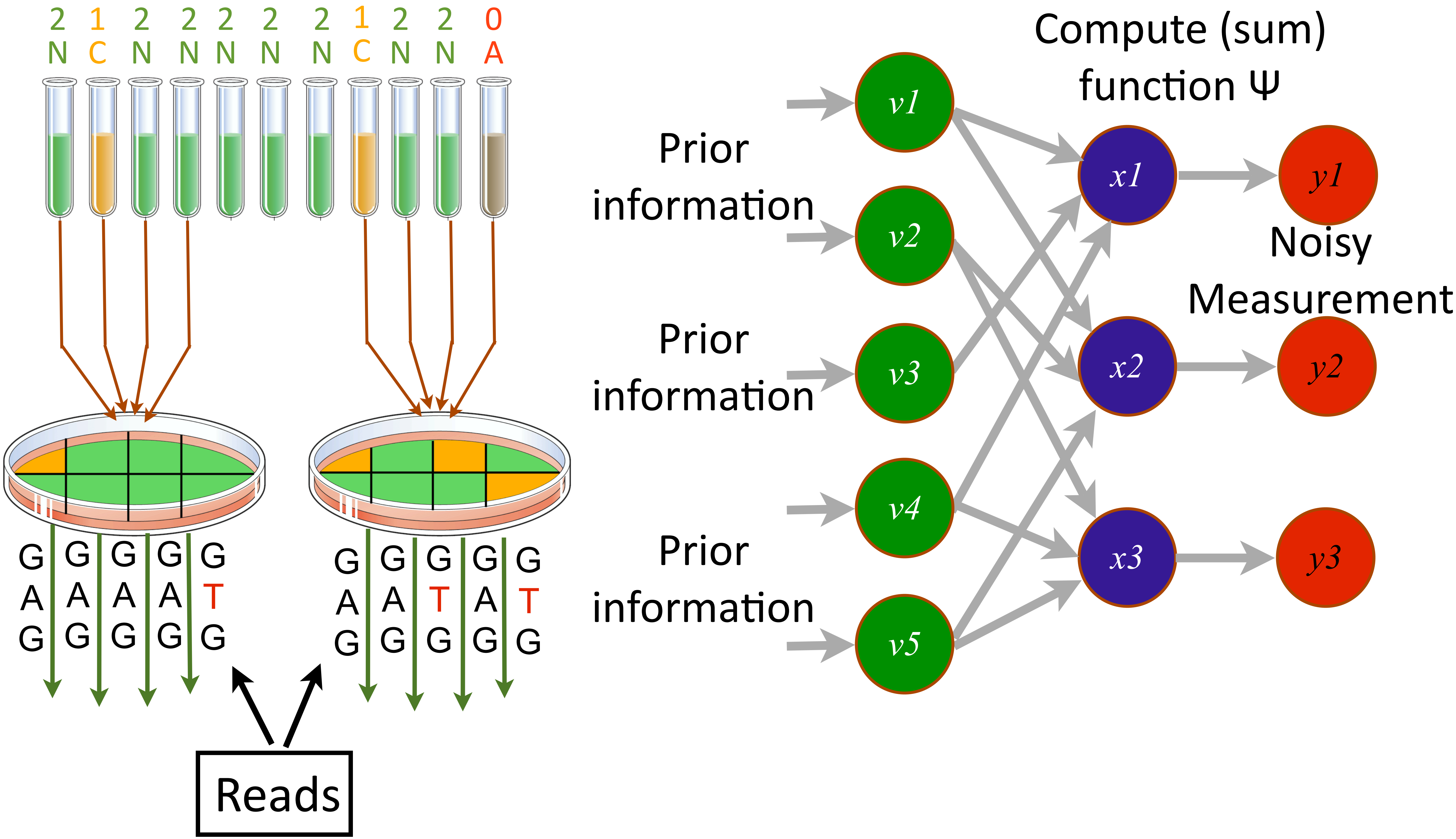


Pooling designs as a sparse graphical model

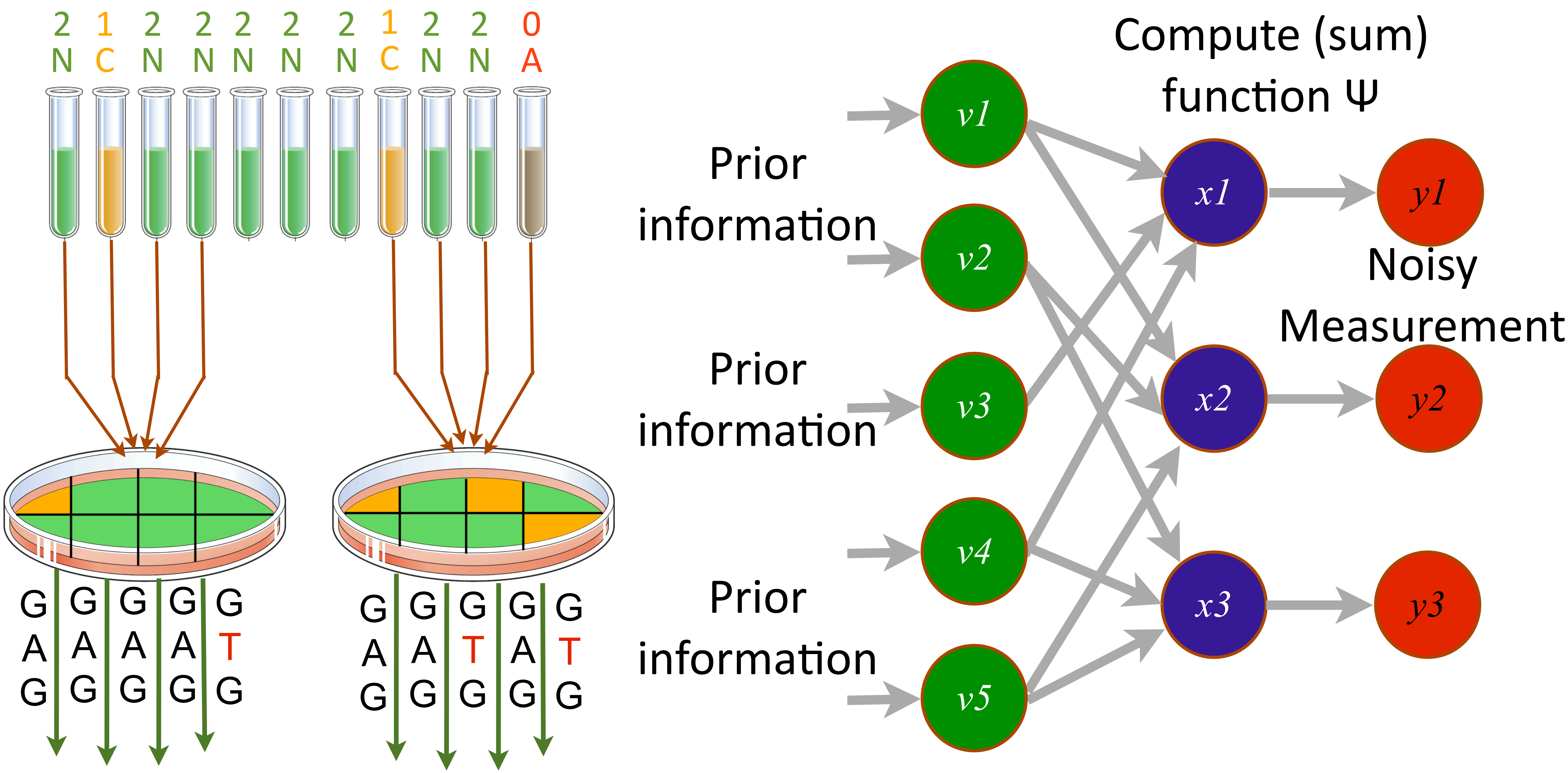




Pooling designs as a sparse graphical model



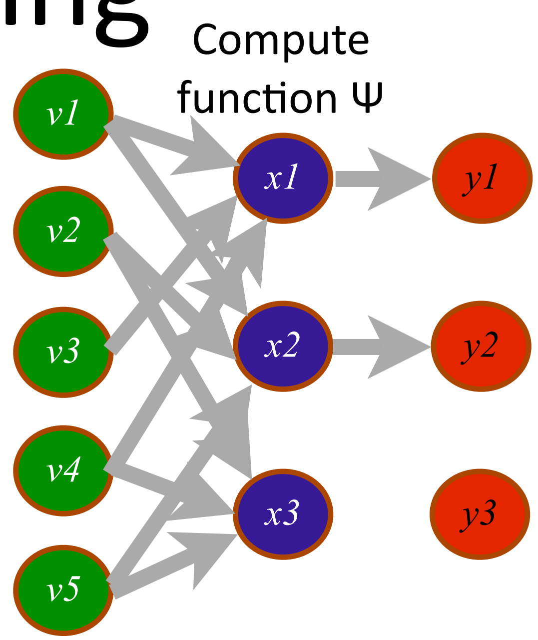
Pooling designs as a sparse graphical model



Sparse regular pooling designs due to practical constraints

Connections to many interesting problems/applications

- Group Testing
- Compressed sensing



- Sensor networks
- LT codes / LDGM codes
- Multi-user detection
- Modeling the olfactory system

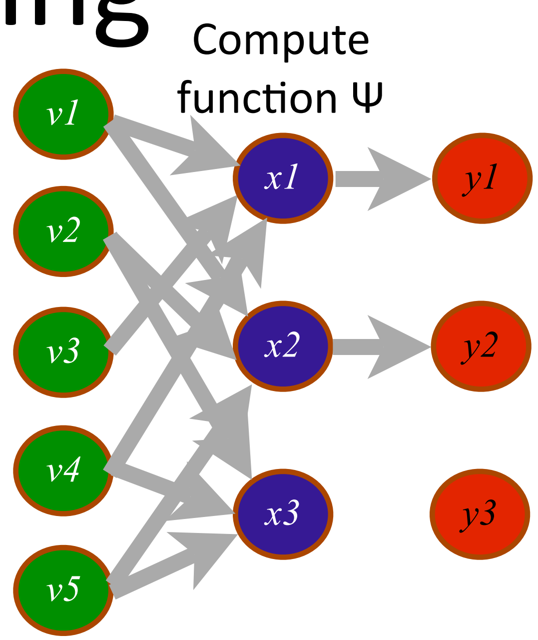
Connections to many interesting problems/applications

- Group Testing

Ψ is OR function

Input/Output binary

- Compressed sensing



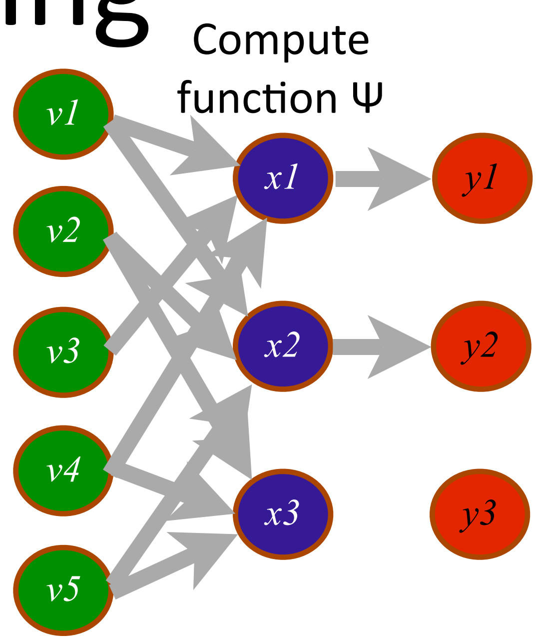
- Sensor networks
- LT codes / LDGM codes
- Multi-user detection
- Modeling the olfactory system

Connections to many interesting problems/applications

- Group Testing

Ψ is OR function
Input/Output binary

- Compressed sensing



- Sensor networks
- LT codes / LDGM codes
- Multi-user detection
- Modeling the olfactory system

- General functions Ψ
- Non-binary inputs and outputs

- Info. Theoretic
- Focus on noise
- Graph Structure
- Probabilistic priors

Rate R = $\frac{\text{\# of inputs}}{\text{\# of measurements}}$

The capacity of sparse, regular measurements

Theorem : A rate R is **achievable** (for an allowable distortion *D*) if, Rate R = # of inputs / # of measurements

$$\text{Rate } R = \frac{\text{\# of inputs}}{\text{\# of measurements}}$$

The capacity of sparse, regular measurements

Theorem : A rate R is **achievable** (for an allowable distortion *D*) if,

$$R < C_{LB}(D) = \min_{\lambda : Dis(\lambda) > D} \frac{T(\lambda)}{[H(\lambda) - H(\gamma)]}$$

Rate R = # of inputs / # of measurements

The capacity of sparse, regular measurements

Theorem : A rate R is **achievable** (for an allowable distortion *D*) if,

$$R < C_{LB}(D) = \min_{\lambda : Dis(\lambda) > D} \frac{T(\lambda)}{[H(\lambda) - H(\gamma)]}$$

For sparse, regular measurement structures

$$T(\lambda) = -H(Y|V) + cH(\lambda) + \inf_{\kappa \,:\, \text{sparse graph constraints}} -H(\kappa) - \sum_{a,b,o} \kappa(a,b) P_{y|v}(o|a) \log P_{y|v}(o|b)$$

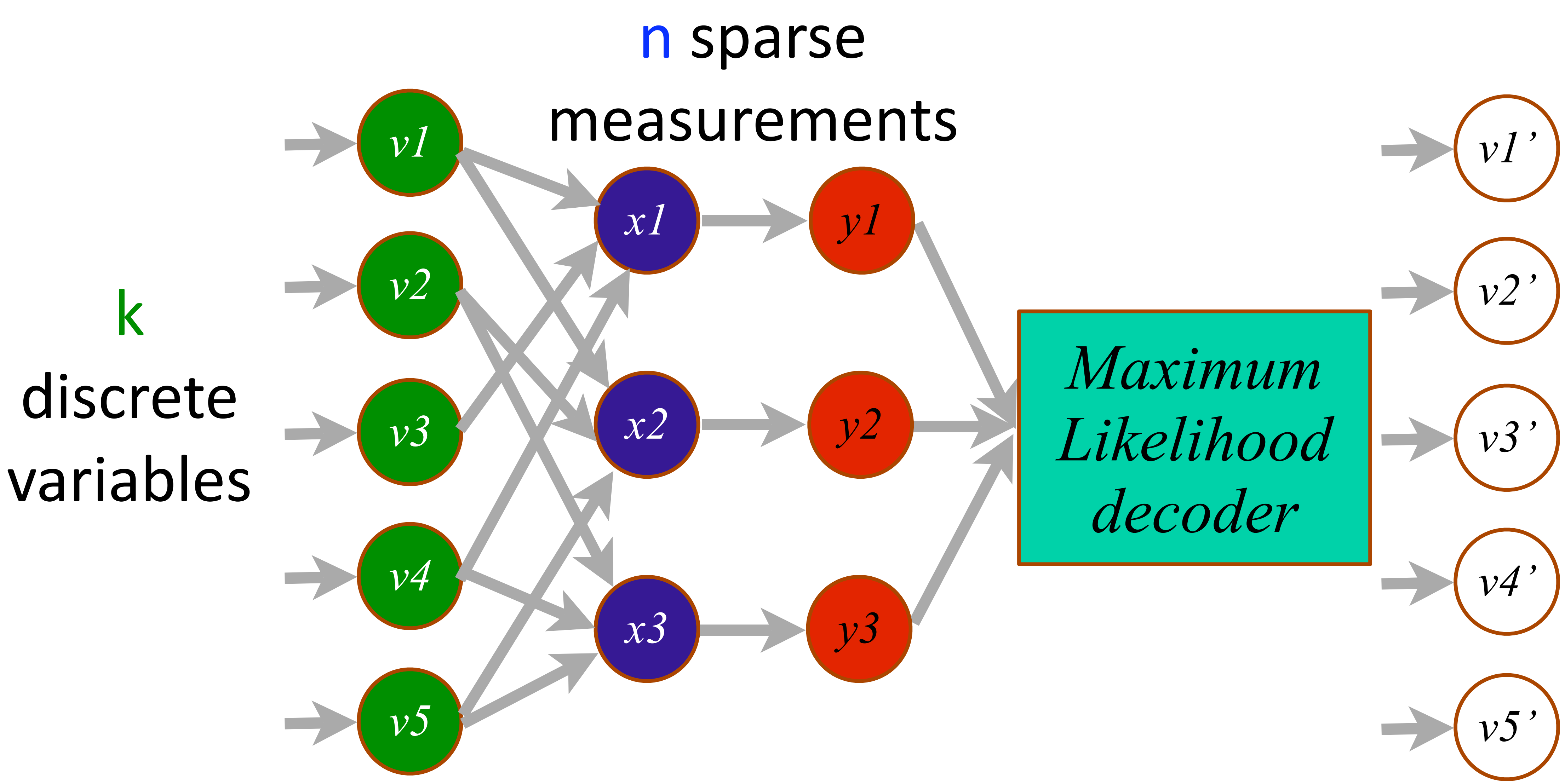
Outline

- Application : Pooling Designs for genetic screening
- Related Problems
- Information theoretic formulation
- Intuition
- Illustration of the result

Analogs : channel coding + rate distortion

$$\text{Rate } R = k/n = \frac{\#\text{ of inputs}}{\#\text{ of measurements}}$$

Distortion = (1/k) Hamming Distance(v,v')



An Information theory for measurements

Rate $R = k/n = \frac{\text{\# of inputs}}{\text{\# of measurements}}$

Distortion = (1/k) Hamming Distance(v,v')

Error if Distortion > D

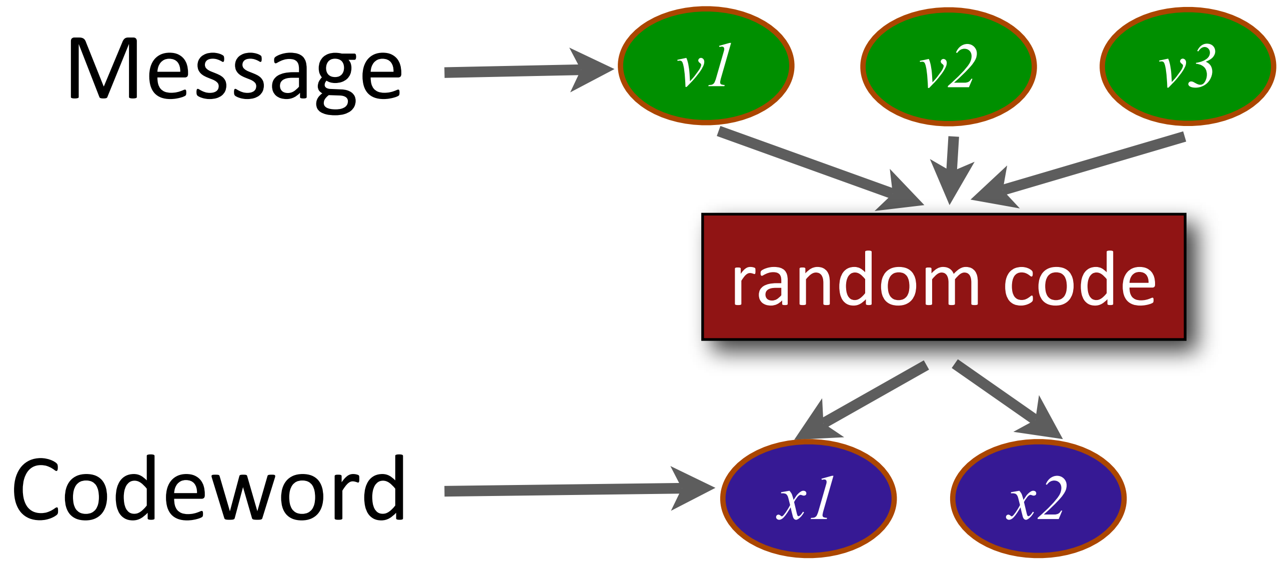
Capacity : C(D) : Maximum R such that $\Pr(\text{Error}) \to 0$ as $n \to \infty$

We lower bound the Capacity : $C_{LB}(D)$

Outline

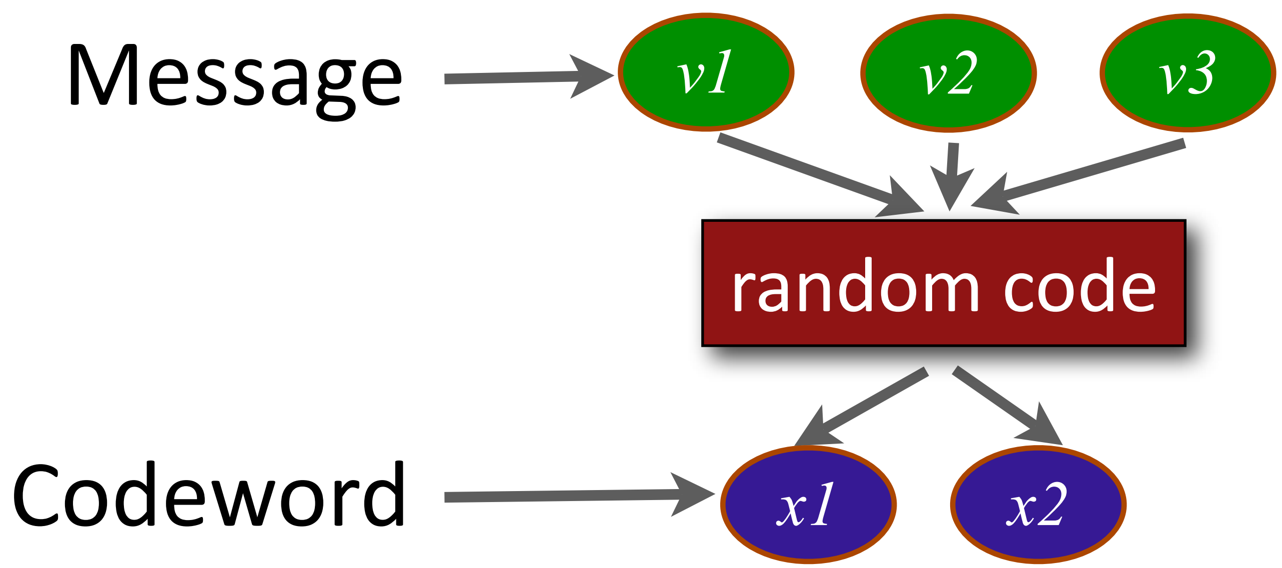
- Application : Pooling Designs for genetic screening
- Related Problems
- Information theoretic formulation
- Intuition
- Illustration of the result

Insight : Parallels to coding theory + rate distortion



Random Coding methods in channel coding

Insight : Parallels to coding theory + rate distortion



Random Coding methods in channel coding

Can we develop a random measurement argument ?

A proof using random measurements

Random measurement configuration

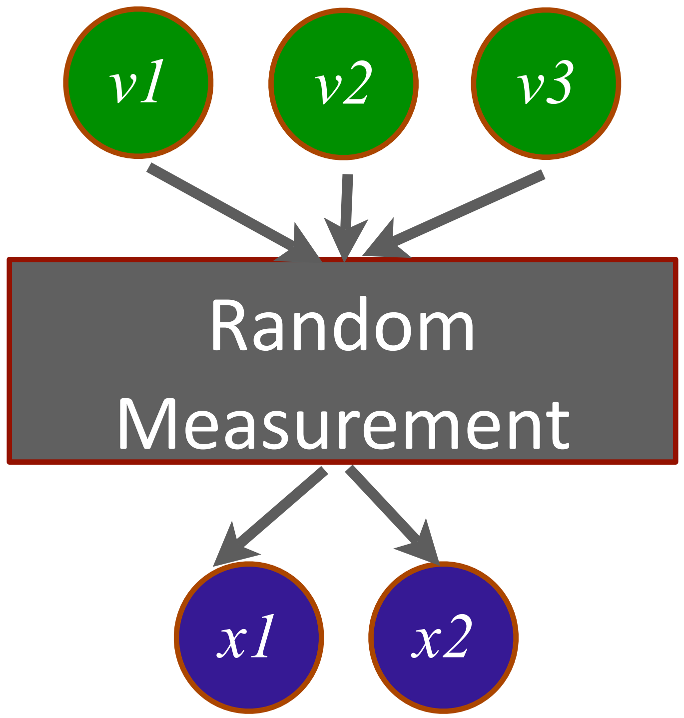
→ generates codebook

Random
Measurement

A proof using random measurements

Random measurement configuration

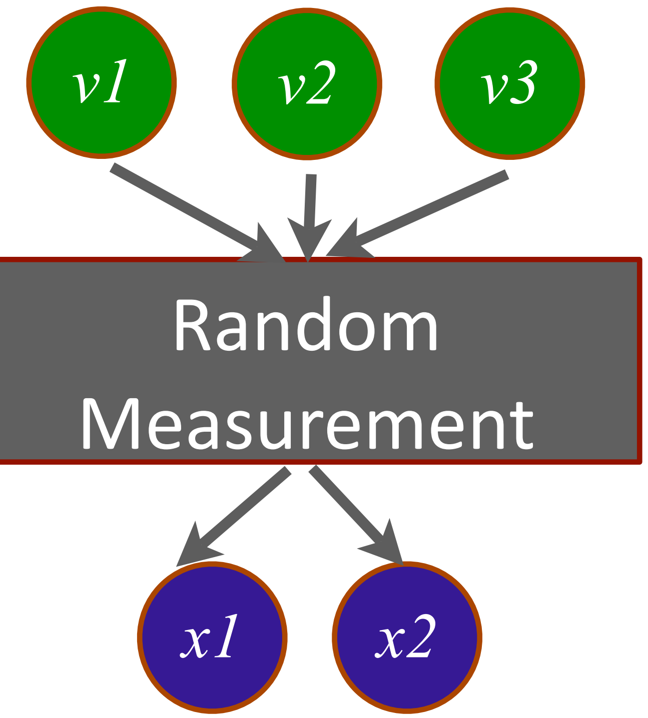
→ generates codebook

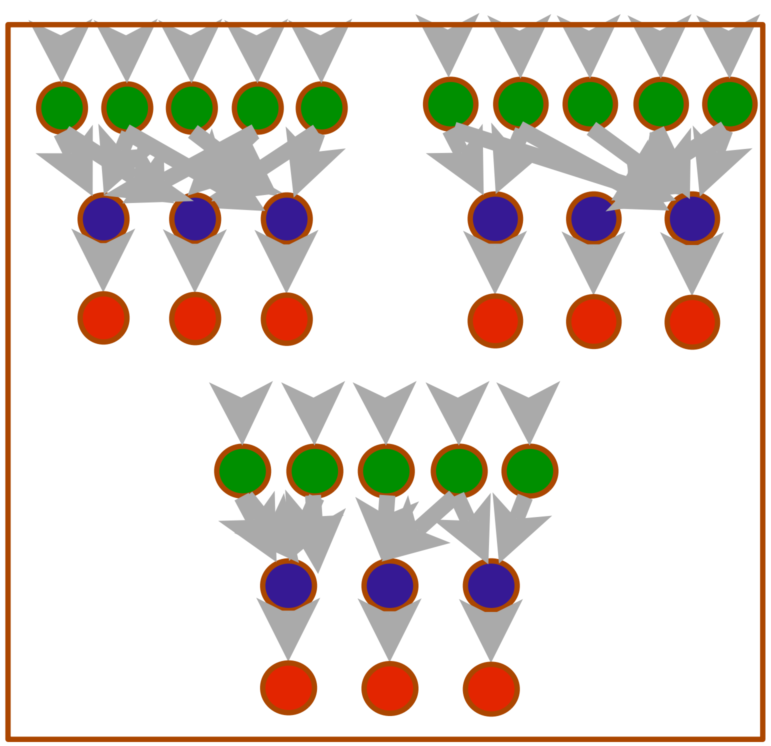


A proof using random measurements

Random measurement configuration

→ generates codebook





Average Error

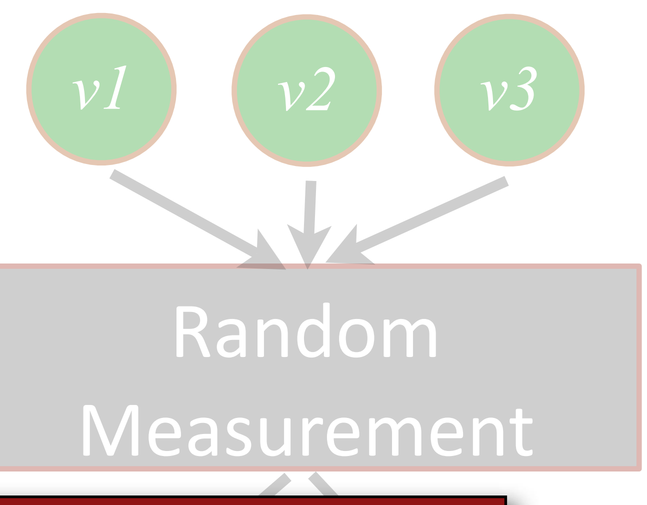
If average error → 0, then

for some configuration error → 0

A proof using random measurements

Random measurement configuration

→ generates codebook



Choose measurement structure not codewords!

If average error → 0, then

for some configuration error → 0

Union Bounding : Gallager-Fano bounding technique

$$
\begin{aligned}
Pr(\text{error}|\mathbf{v} \text{ is true}) &= Pr(\text{Decode to } \mathbf{v}' \text{ s.t distortion}(\mathbf{v}, \mathbf{v}') > D \mid \mathbf{v} \text{ is true}) \\
&\leq \sum_{\mathbf{v}' \neq \mathbf{v}} Pr(\text{Decode to } \mathbf{v}' \mid \mathbf{v} \text{ is true})
\end{aligned}
$$

Exponential number of terms !!!

Union Bounding : Gallager-Fano bounding technique

$$Pr(\text{error}|\mathbf{v} \text{ is true}) = Pr(\text{Decode to } \mathbf{v}' \text{ s.t distortion}(\mathbf{v},\mathbf{v}') > D \mid \mathbf{v} \text{ is true})$$
$$\leq \sum_{\mathbf{v}' \neq \mathbf{v}} Pr(\text{Decode to } \mathbf{v}' \mid \mathbf{v} \text{ is true})$$

Exponential number of terms !!!

Group terms into polynomial number of groups *g* using symmetry

$$Pr(\text{error}|\mathbf{v} \text{ is true}) \leq \sum_{g} (\text{number of } \mathbf{v}' \text{ in } g) Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true })$$
$$\leq |g| \max_{g} (\text{number of } \mathbf{v}' \text{ in } g) Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true})$$

Union Bounding : Gallager-Fano bounding technique

$$
\begin{aligned}
Pr(\text{error}|\mathbf{v}\text{ is true}) &= Pr(\text{Decode to }\mathbf{v}'\text{ s.t distortion}(\mathbf{v},\mathbf{v}') > D \mid \mathbf{v}\text{ is true}) \\
&\leq \sum_{\mathbf{v}'\neq\mathbf{v}} Pr(\text{Decode to }\mathbf{v}' \mid \mathbf{v}\text{ is true})
\end{aligned}
$$

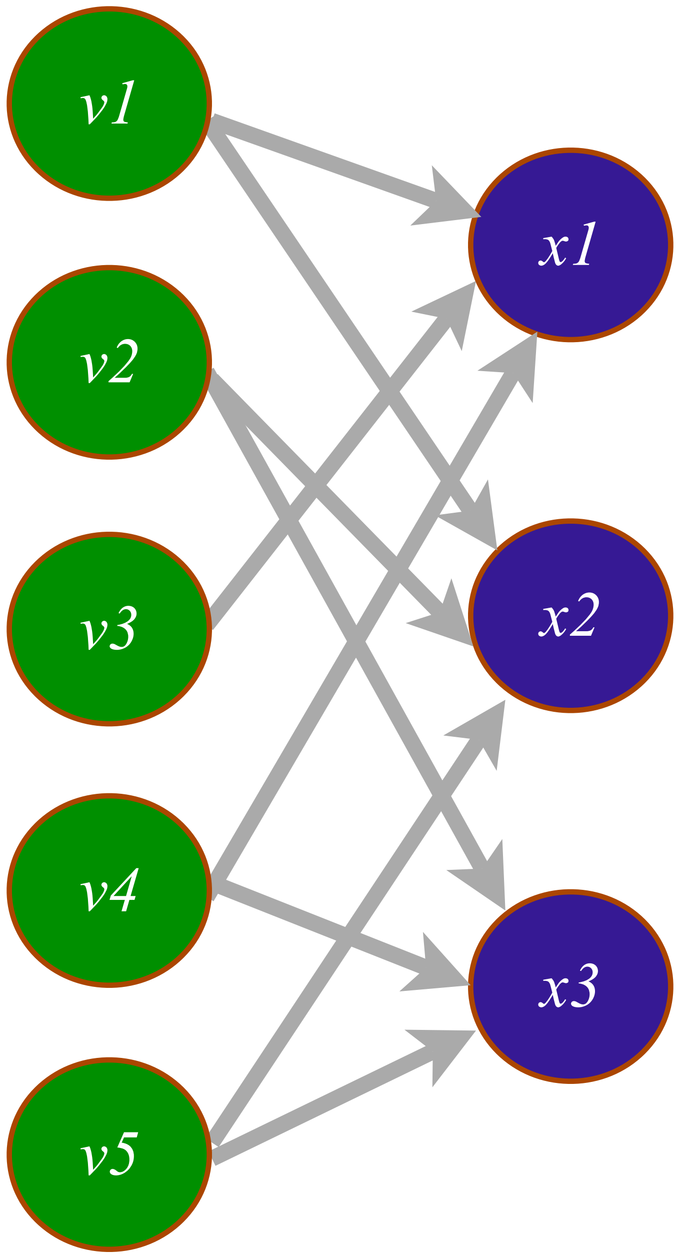
Exponential number of terms !!!

Group terms into polynomial number of groups *g* using symmetry

$$
\begin{aligned}
Pr(\text{error}|\mathbf{v}\text{ is true}) &\leq \sum_{g} (\text{number of }\mathbf{v}'\text{ in }g) Pr(\text{Decode to }\mathbf{v}'\text{ in }g \mid \mathbf{v}\text{ is true }) \\
&\leq |g| \max_{g} (\text{number of }\mathbf{v}'\text{ in }g) Pr(\text{Decode to }\mathbf{v}'\text{ in }g \mid \mathbf{v}\text{ is true})
\end{aligned}
$$

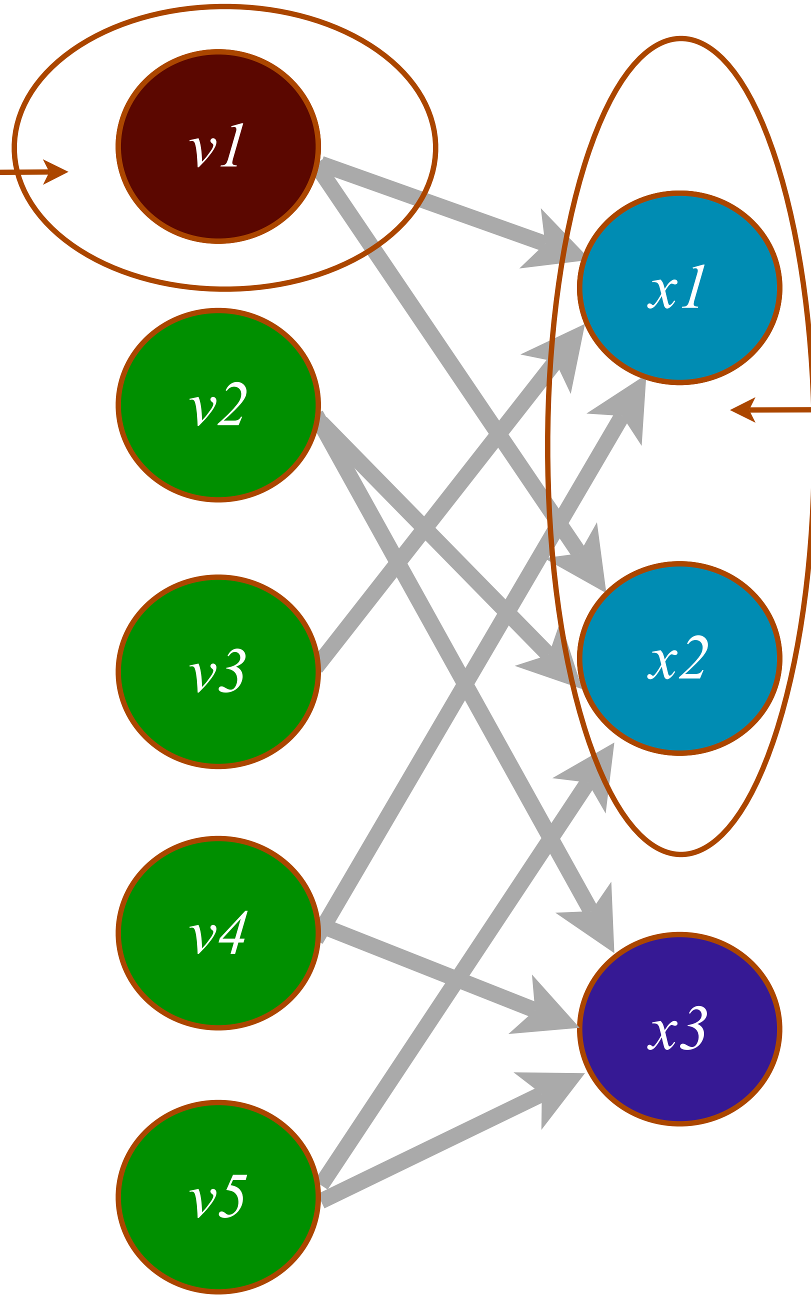
Easy when the codewords are i.i.d !

Complication - Loss of Symmetry : Non-i.i.d codewords



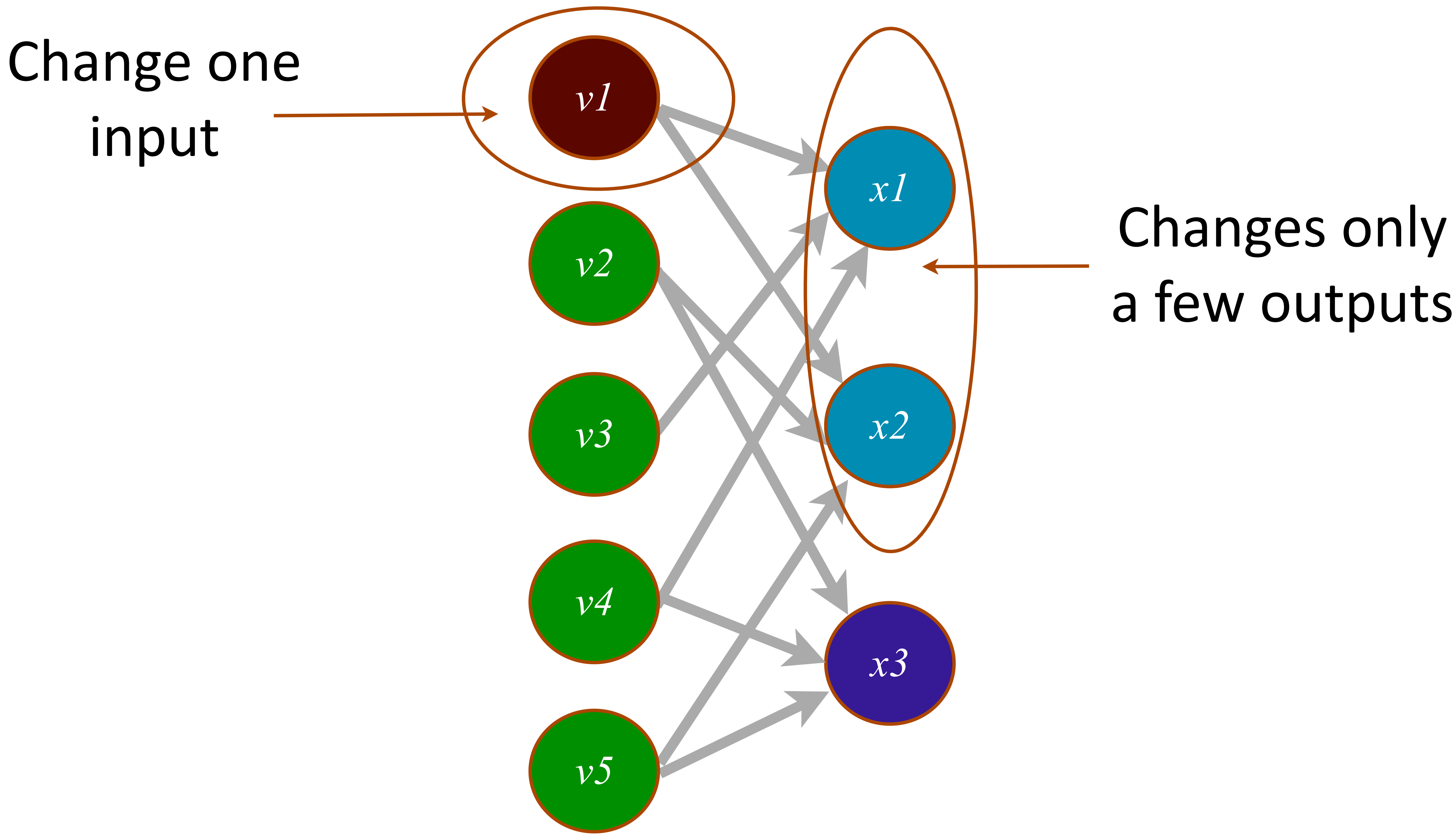
Complication - Loss of Symmetry : Non-i.i.d codewords

Change one input



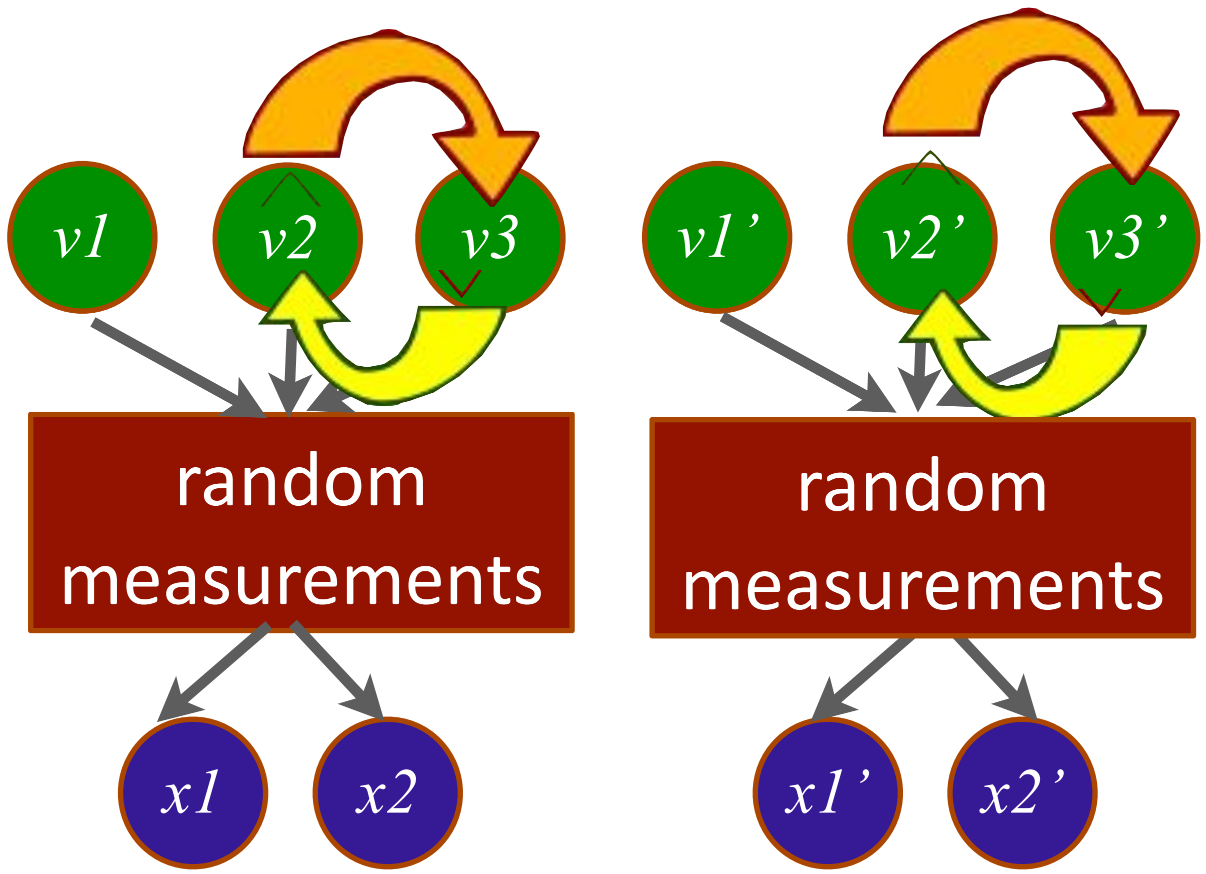
Changes only a few outputs

Complication - Loss of Symmetry : Non-i.i.d codewords

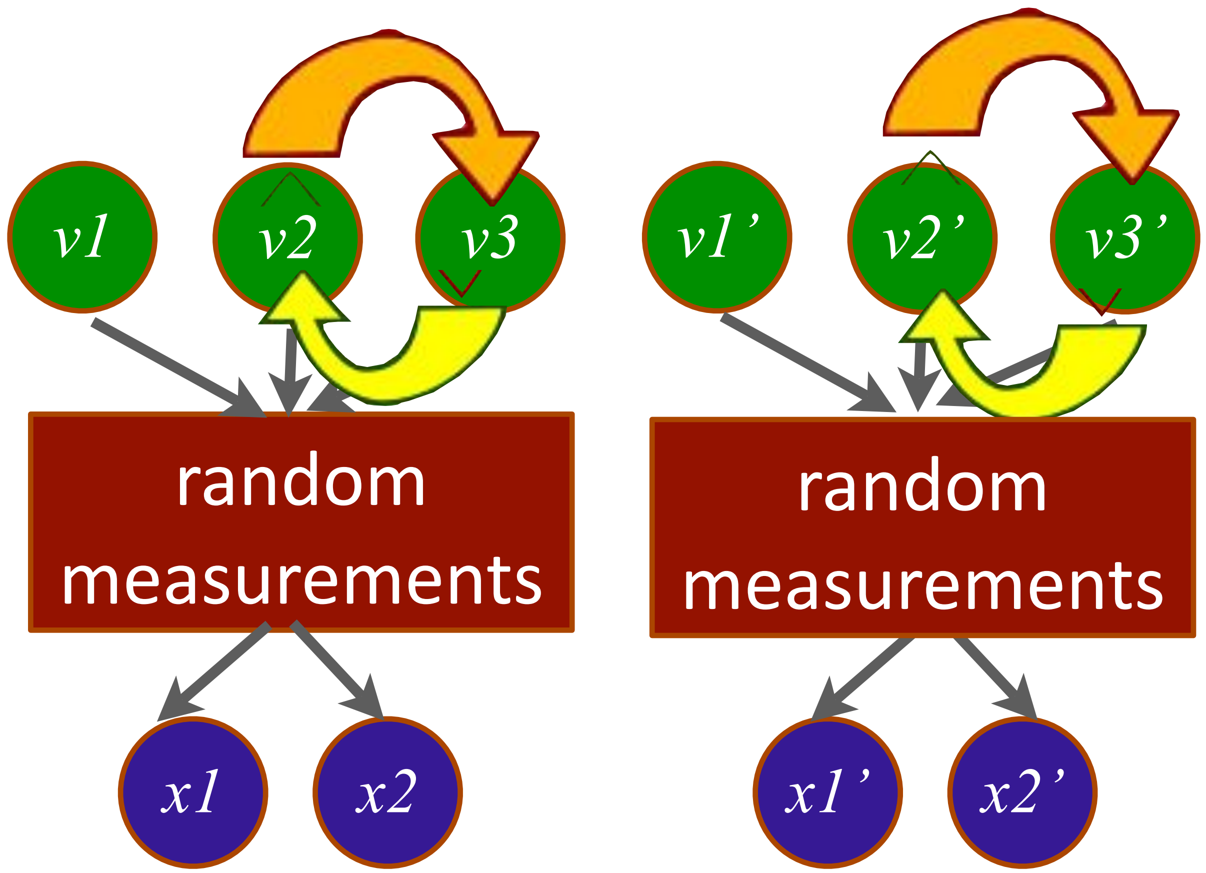


Classical proof doesn't work !

Reintroducing Symmetry: Permutation invariant measurement ensembles

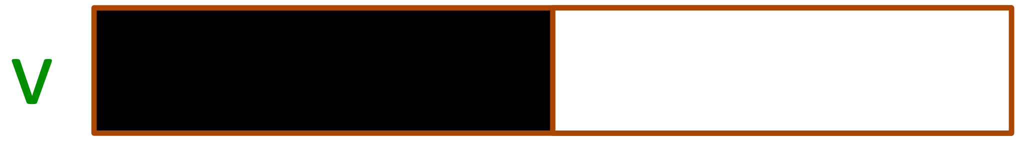


Reintroducing Symmetry: Permutation invariant measurement ensembles



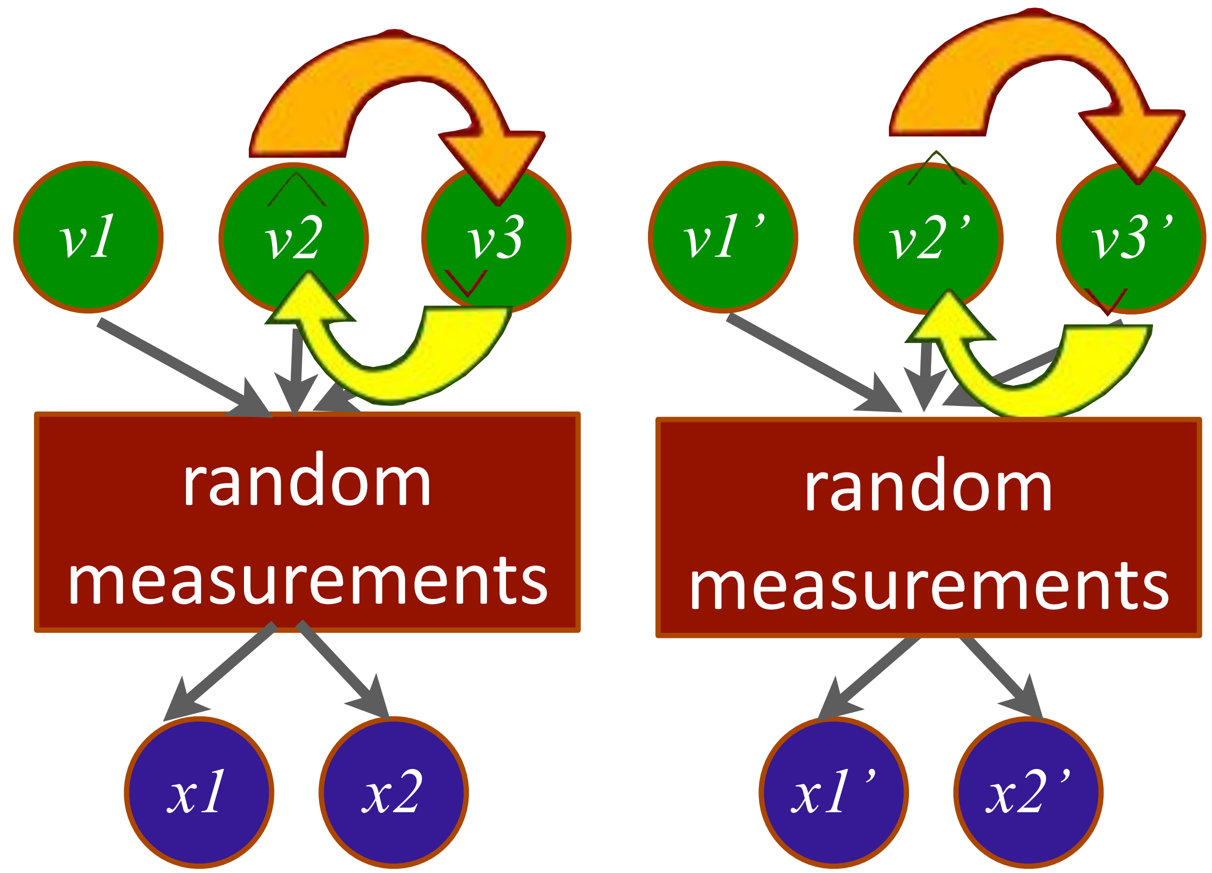
$$P_{\mathbf{v}}(\mathbf{X})$$

depends only on the type γ of $\mathbf{v}$



Υ [1/2 1/2]

Reintroducing Symmetry: Permutation invariant measurement ensembles

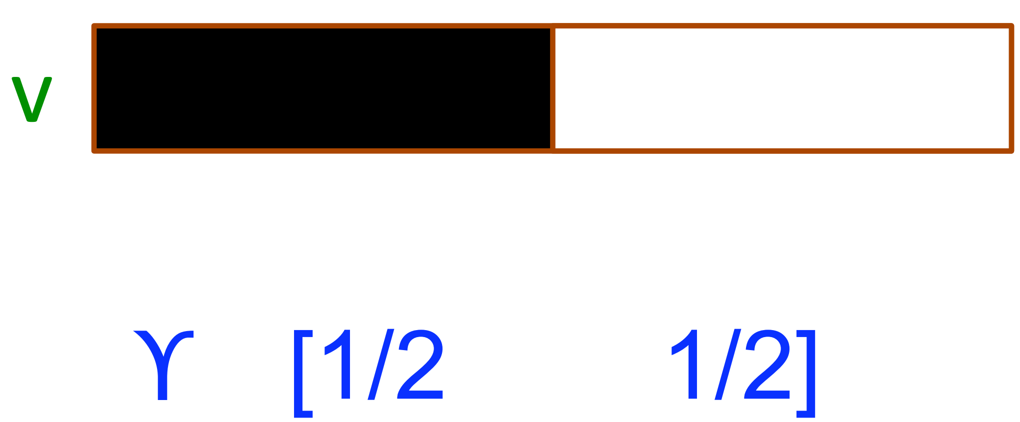


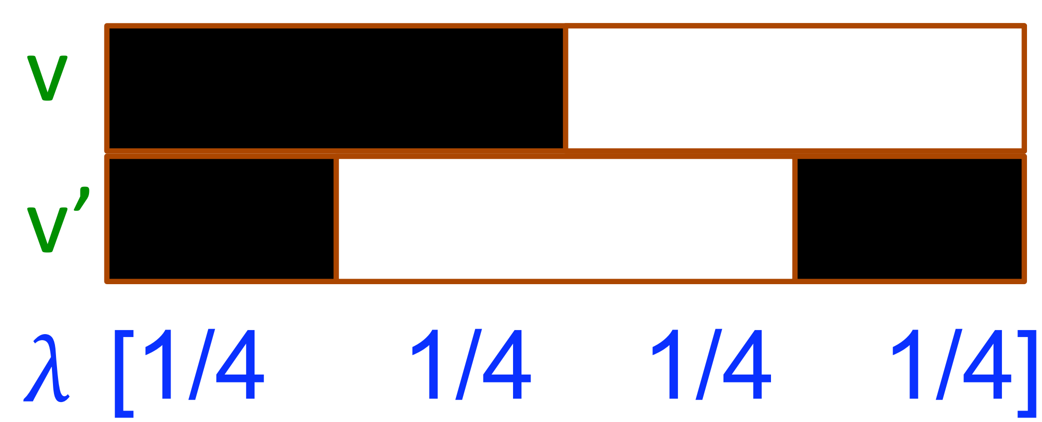
$$P_{\mathbf{v}}(\mathbf{X})$$

depends only on the type γ of $\mathbf{v}$

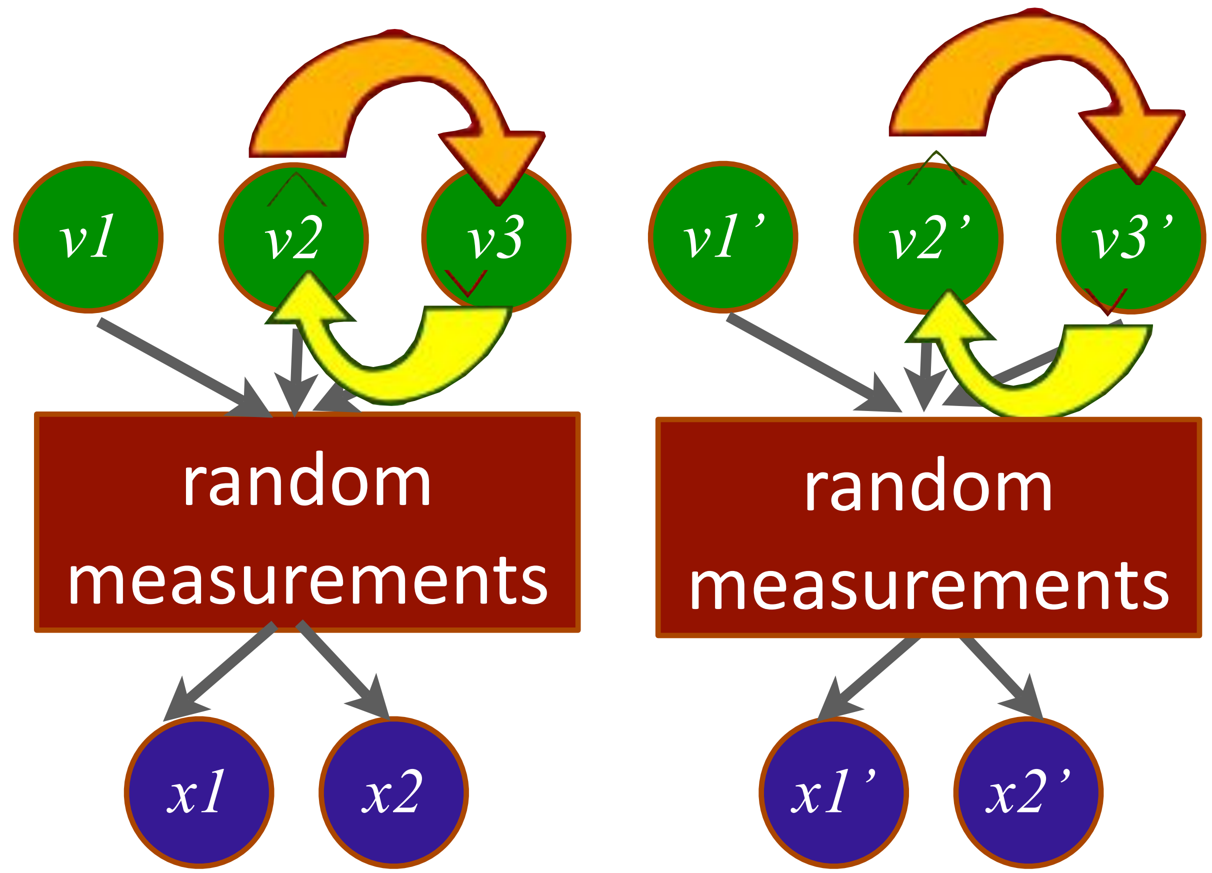
$$Q_{\mathbf{v}',\mathbf{v}}(\mathbf{X}'|\mathbf{X})$$

depends only on the joint type λ of $\mathbf{v}$ and $\mathbf{v}'$





Reintroducing Symmetry: Permutation invariant measurement ensembles

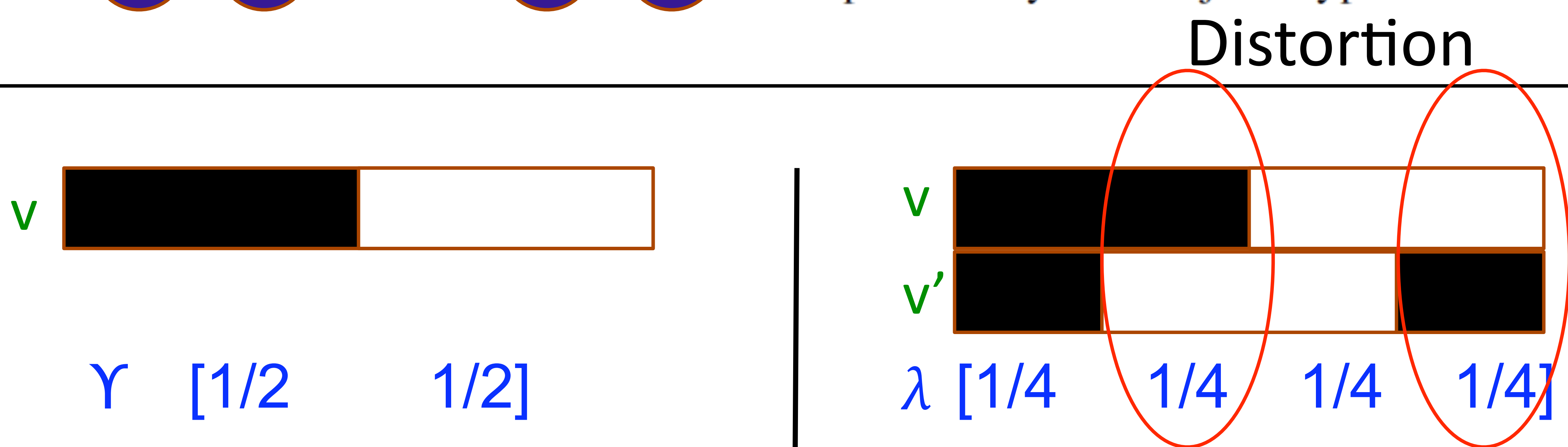


$$P_{\mathbf{v}}(\mathbf{X})$$

depends only on the type γ of $\mathbf{v}$

$$Q_{\mathbf{v}',\mathbf{v}}(\mathbf{X}'|\mathbf{X})$$

depends only on the joint type λ of $\mathbf{v}$ and $\mathbf{v}'$



Using the symmetry

$$\begin{aligned} Pr(\text{error}|\mathbf{v} \text{ is true}) &\leq \sum_g (\text{number of } \mathbf{v}' \text{ in } g) Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true }) \\ &\leq |g| \max_g (\text{number of } \mathbf{v}' \text{ in } g) Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true}) \end{aligned}$$

Using the symmetry

$$
\begin{aligned}
Pr(\text{error}|\mathbf{v} \text{ is true}) &\leq \sum_g (\text{number of } \mathbf{v}' \text{ in } g) Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true })\\
&\leq |g| \max_g (\text{number of } \mathbf{v}' \text{ in } g) Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true})
\end{aligned}
$$

$$Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true})$$

Depends only on $P_{\mathbf{v}}(\mathbf{X})$ and $Q_{\mathbf{v}',\mathbf{v}}(\mathbf{X}'|\mathbf{X})$ i.e γ and λ

Using the symmetry

$$
\begin{aligned}
Pr(\text{error}|\mathbf{v} \text{ is true}) &\leq \sum_g (\text{number of } \mathbf{v}' \text{ in } g) Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true }) \\
&\leq |g| \max_g (\text{number of } \mathbf{v}' \text{ in } g) Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true})
\end{aligned}
$$

$$Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true})$$

Depends only on $P_{\mathbf{v}}(\mathbf{X})$ and $Q_{\mathbf{v}',\mathbf{v}}(\mathbf{X}'|\mathbf{X})$ i.e γ and λ

Use the joint types λ as the symmetry groups g

$$= Pr(\text{Decode to } \mathbf{v}' \text{ at } \lambda \mid \mathbf{v} \text{ is true})$$

Using the symmetry

$$
\begin{aligned}
Pr(\text{error}|\mathbf{v} \text{ is true}) &\leq \sum_g (\text{number of } \mathbf{v}' \text{ in } g) Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true }) \\
&\leq |g| \max_g (\text{number of } \mathbf{v}' \text{ in } g) Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true})
\end{aligned}
$$

$$Pr(\text{Decode to } \mathbf{v}' \text{ in } g \mid \mathbf{v} \text{ is true})$$

Depends only on $P_{\mathbf{v}}(\mathbf{X})$ and $Q_{\mathbf{v}',\mathbf{v}}(\mathbf{X}'|\mathbf{X})$ i.e γ and λ

Use the joint types λ as the symmetry groups g

$$= Pr(\text{Decode to } \mathbf{v}' \text{ at } \lambda \mid \mathbf{v} \text{ is true})$$

$$Pr(\text{error}|\mathbf{v} \text{ is true}) \leq |\lambda| \max_\lambda (\text{number of } \mathbf{v}' \text{ at } \lambda) Pr(\text{Decode to } \mathbf{v}' \text{ at } \lambda \mid \mathbf{v} \text{ is true})$$

Only a polynomial number of joint types λ

Some combinatorics and some large deviations

$$\text{Number of } \mathbf{v}' \text{ at } \lambda \leq 2^{k[H(\lambda) - H(\gamma)]}$$

Method of types

Some combinatorics and some large deviations

$$\text{Number of } \mathbf{v}' \text{ at } \lambda \leq 2^{k[H(\lambda)-H(\gamma)]}$$

Method of types

$$Pr(\text{Decode to } \mathbf{v}' \text{ at } \lambda \mid \mathbf{v} \text{ is true}) = Pr\left[P(\mathbf{y}|\mathbf{v}') > P(\mathbf{y}|\mathbf{v})\right]$$

Some combinatorics and some large deviations

$$\text{Number of } \mathbf{v}' \text{ at } \lambda \leq 2^{k[H(\lambda)-H(\gamma)]}$$

Method of types

$$Pr(\text{Decode to } \mathbf{v}' \text{ at } \lambda \mid \mathbf{v} \text{ is true}) = Pr\left[P(\mathbf{y}|\mathbf{v}') > P(\mathbf{y}|\mathbf{v})\right]$$

Large deviations for sparse regular measurements

$$\frac{1}{N}\log\left(\frac{P(\mathbf{y}|\mathbf{v})}{E\left[P(\mathbf{y}|\mathbf{v})\right]}\right) \to T(\lambda)$$

Some combinatorics and some large deviations

$$\text{Number of } \mathbf{v}' \text{ at } \lambda \leq 2^{k[H(\lambda)-H(\gamma)]}$$

Method of types

$$Pr(\text{Decode to } \mathbf{v}' \text{ at } \lambda \mid \mathbf{v} \text{ is true}) = Pr\left[P(\mathbf{y}|\mathbf{v}') > P(\mathbf{y}|\mathbf{v})\right]$$

Large deviations for sparse regular measurements

$$\frac{1}{N}\log\left(\frac{P(\mathbf{y}|\mathbf{v})}{E\left[P(\mathbf{y}|\mathbf{v})\right]}\right) \to T(\lambda)$$

Introduce tilting distributions

$$Pr(\text{Decode to } \mathbf{v}' \text{ at } \lambda \mid \mathbf{v} \text{ is true}) \leq 2^{-nT(\lambda)}$$

Result : Lower Bound on Capacity

Rate R=k/n= $\dfrac{\text{\# of inputs}}{\text{\# of measurements}}$

A rate R is **achievable** for a joint type λ is

$$R < \frac{T(\lambda)}{[H(\lambda) - H(\gamma)]}$$

Result : Lower Bound on Capacity

Rate R=k/n= $\frac{\text{\# of inputs}}{\text{\# of measurements}}$

A rate R is **achievable** for a joint type λ is

$$R < \frac{T(\lambda)}{[H(\lambda) - H(\gamma)]}$$

A rate R is **achievable** for a distortion *D* if

$$R < C_{LB}(D) = \min_{\lambda: Dis(\lambda) > D} \frac{T(\lambda)}{[H(\lambda) - H(\gamma)]}$$

Result : Lower Bound on Capacity

Rate R=k/n= # of inputs / # of measurements

For sparse, regular measurement structures

$$T(\lambda) = -H(Y|V) + cH(\lambda) + \inf_{\kappa\,:\,\text{sparse graph constraints}} -H(\kappa) - \sum_{a,b,o} \kappa(a,b) P_{y|v}(o|a) \log P_{y|v}(o|b)$$

A rate R is **achievable** for a distortion *D* if

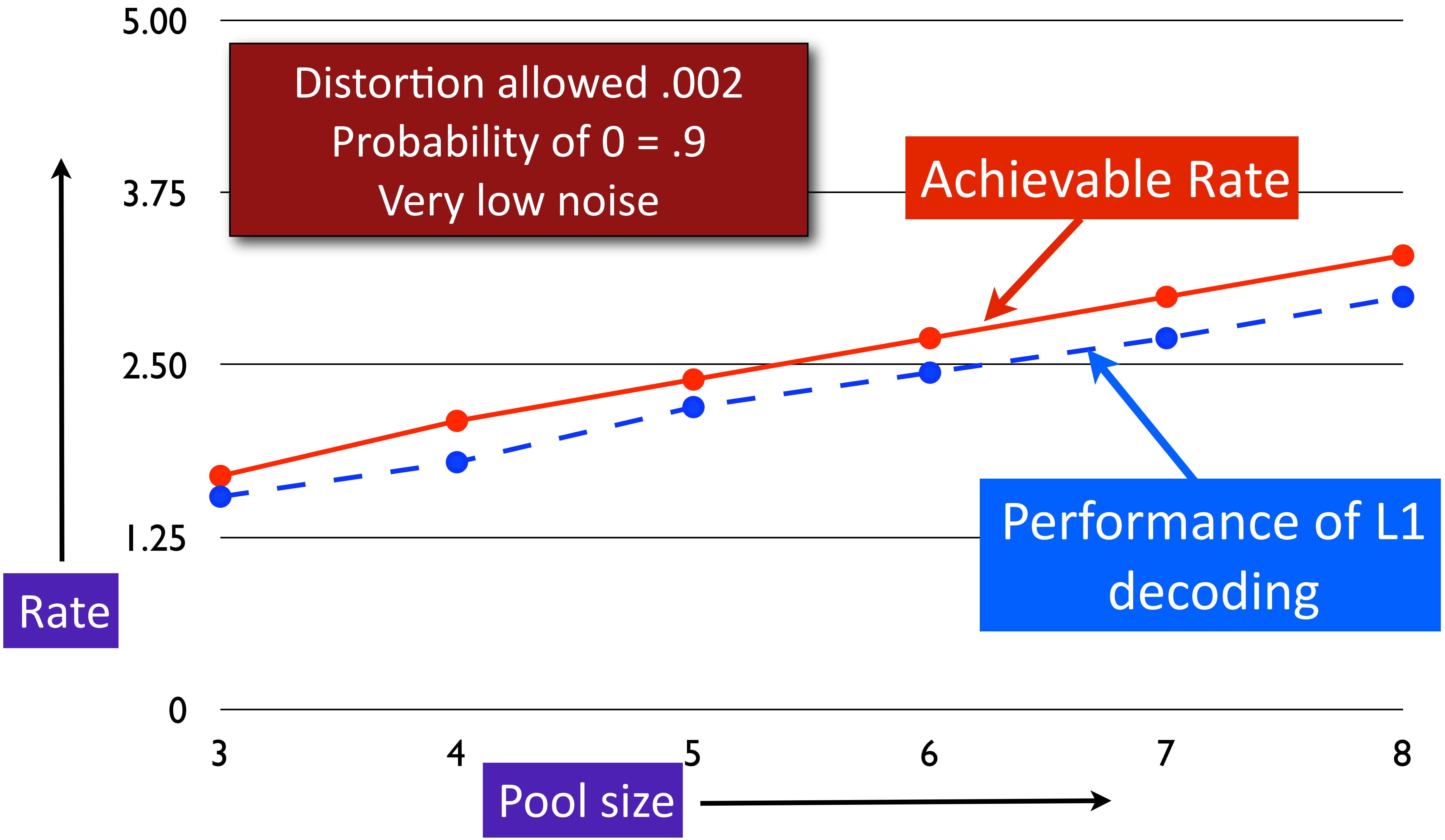
$$R < C_{LB}(D) = \min_{\lambda : Dis(\lambda) > D} \frac{T(\lambda)}{[H(\lambda) - H(\gamma)]}$$

Outline

- Application : Pooling Designs for genetic screening
- Related Problems
- Information theoretic formulation
- Intuition
- Illustration of the result

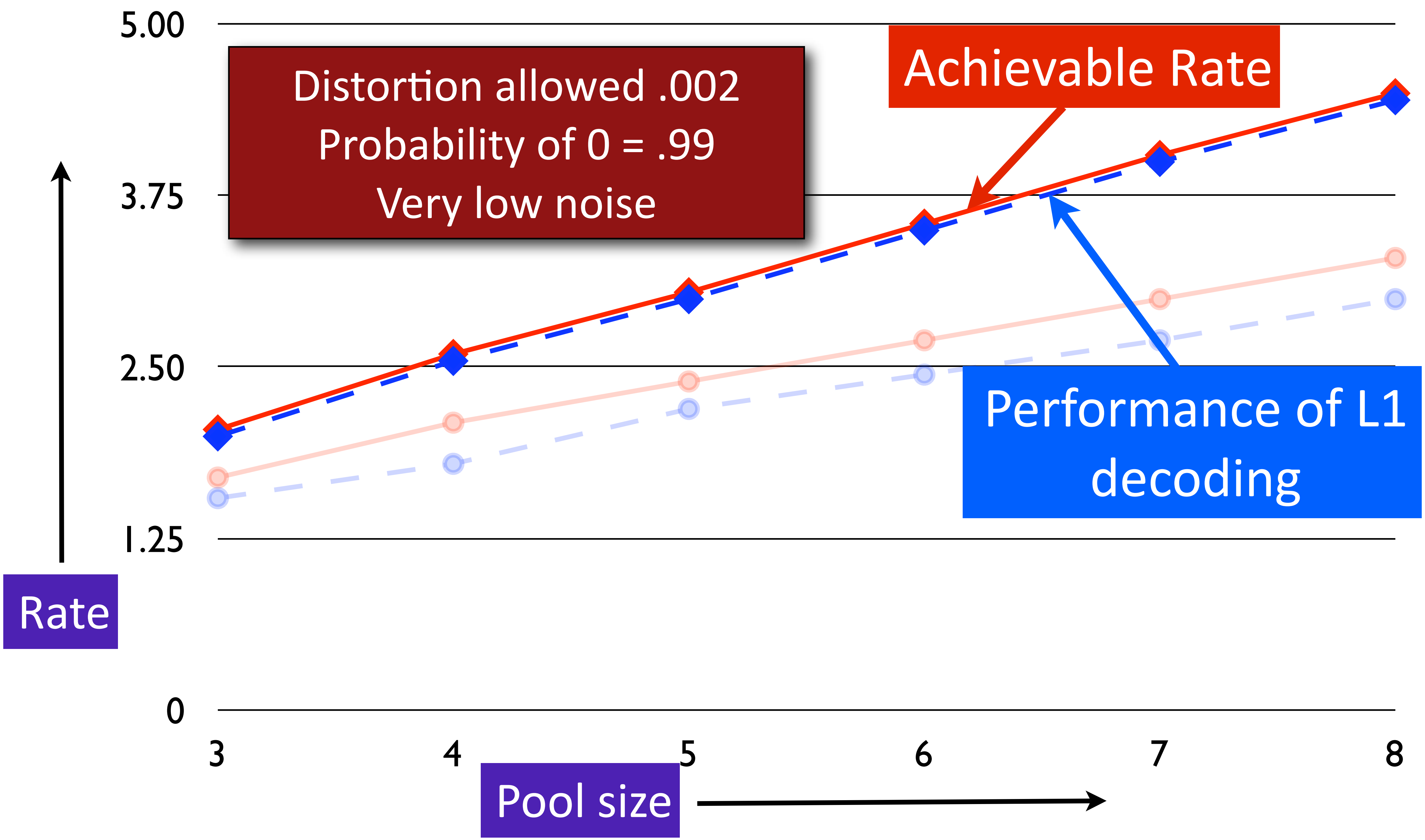
Capacity predicts performance of practical decoder

Rate R=k/n= # of inputs / # of measurements

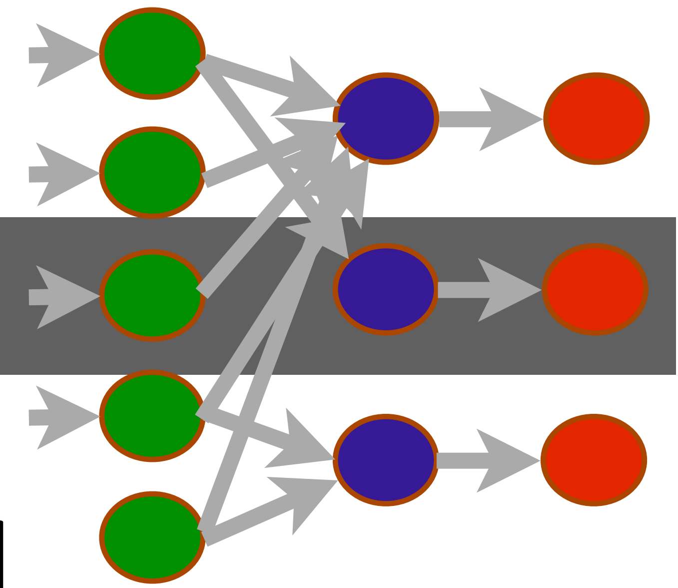


Excellent agreement for very sparse priors

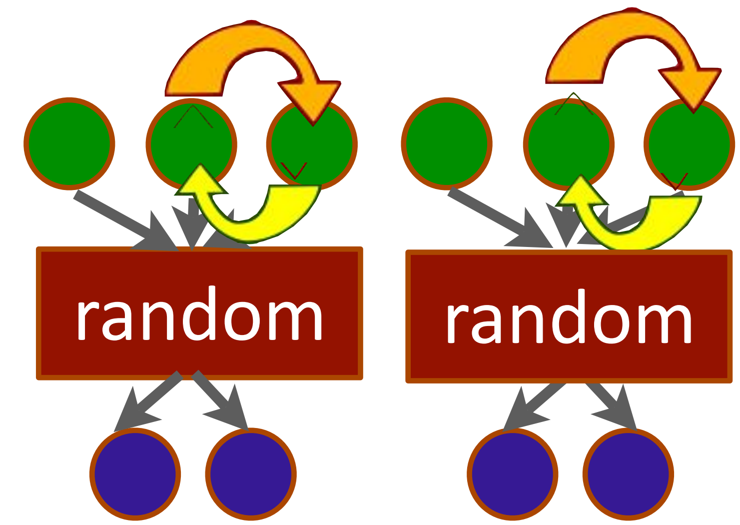
Rate R=k/n= $\frac{\text{\# of inputs}}{\text{\# of measurements}}$



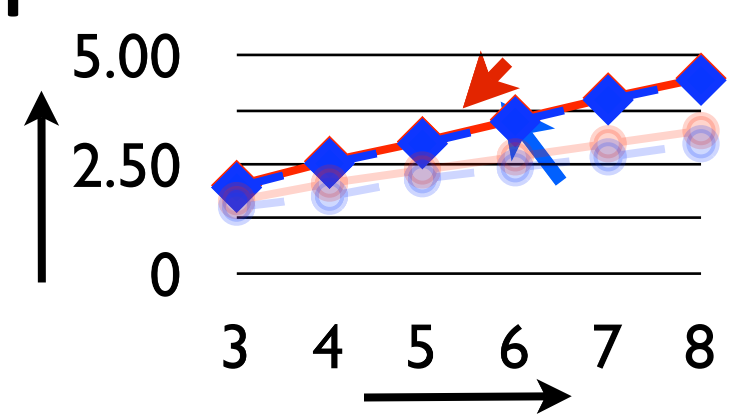
Insights



1. Analog between measurements and communication/coding



2. Random Measurements Argument
3. Symmetry through permutation invariance
4. Method of Types + Large Deviation
5. Tightness/Relevance of the result



More combinatorics

Suppose all pools are of size c

All inputs are in alphabet of size L

δ	Type/distribution over L^c with marginals Υ
κ	Joint type/distribution over (L^c, L^c) with marginals λ

More combinatorics

Suppose all pools are of size c

All inputs are in alphabet of size L

δ	Type/distribution over L^c with marginals Υ
κ	Joint type/distribution over (L^c, L^c) with marginals λ

$$P_m(\vec{\mathbf{Z}}) = P^{\vec{\gamma},\vec{\delta}} = \begin{cases} 2^{-ncH(\vec{\gamma})} & \vec{\delta} \text{ consistent with } \vec{\gamma} \\ 0 & \text{otherwise} \end{cases}$$

$$P_{m,m'}(\vec{\mathbf{Z}}, \vec{\mathbf{Z}}') = P^{\vec{\lambda},\vec{\kappa}} = \begin{cases} 2^{-ncH(\vec{\lambda})} & \vec{\kappa} \text{ consistent with } \vec{\lambda} \\ 0 & \text{otherwise} \end{cases}$$

The capacity of sparse, regular measurements

Theorem : A rate R is **achievable** (for an allowable distortion *D*) if,

$$R < C_{LB}(D) = \min_{\lambda : Dis(\lambda) > D} \frac{T(\lambda)}{[H(\lambda) - H(\gamma)]}$$

For sparse, regular measurement structures

$$T(\lambda) = -H(Y|V) + cH(\lambda) + \inf_{\kappa\ :\ \text{sparse graph constraints}} -H(\kappa) - \sum_{a,b,o} \kappa(a,b) P_{y|v}(o|a) \log P_{y|v}(o|b)$$