Uncertainty 3:

State Space Kalman Filters

1 Kalman/Bucy Filter

- Invented by R. E. Kalman in 1960's or so. One year later Kalman and Bucy jointly published the continuous time version.
- A purely mathematical algorithm for estimating the state of a dynamic system based on recursive measurement of noisy data.
- Useful for perception, position estimation, control - generally, any measurement task.
- Prerequisites for use are that corruptive errors must be:
    - Unbiased (have zero mean for all time)
    - Gaussian (have a Gaussian distribution for all time)
    - White (contain all frequencies)
- Measurements may generally be:
    - incomplete: related to some of the variables of interest
    - indirect: related indirectly to the quantities of interest
    - intermittent: available at irregular intervals of time
    - inexact: corrupted by many forms of error
- The state space form has these additional abilities:
    - it can predict dynamic system state independent of measurements.
    - it can easily use rate measurements that are derivatives of required state variables.
    - it can explicitly account for modelling assumptions and disturbances in a more precise way than just "noise".
    - it can identify a system (calibrate parameters) in real-time.

2 State Space Model of a Random Process

• The basic pilosophy is to model the system of interest as a linear dynamic system which is excited by noise and whose sensors are also excited by noise.

• The system model is a matrix linear differential equation. Such a model considers the process to be the result of passing white noise through a system with linear dynamics.

• The **state vector** for a dynamic system is any set of quantities sufficient to completely describe the unforced motion of the system. Given the state at any point in time, the state at any future time can be determined from the control inputs and the system state space model.

• Intuitively, a state vector contains values for all variables in the system up to one order less than the highest order derivative represented in the model. This is, of course, the exact number of initial conditions required to solve a differential equation.

• Time may be considered continuous or discrete and models of one form can converted to the other.

2.1 Continuous Model

• Linear systems model, or **state model** of a random process:

$$\dot{\underline{x}} = F\underline{x} + G\underline{w} \qquad \text{“state” or “process” model}$$

$$\underline{z} = H\underline{x} + \underline{v} \qquad \text{“measurement” or “observation” model}$$

• This is a nice analogy but never used in practice.

2.2 Discrete Model

• If time is considered to be discrete, the process is described in the following form:

$$\underline{x}_{k+1} = \Phi_k \bar{x}_k + \Gamma_k \underline{w}_k$$ "state" or "process" model

$$\underline{z}_k = H_k \bar{\underline{x}}_k + \bar{v}_k$$ "measurement" or "observation" model

• The names and sizes of the vectors and matrices are:

$\underline{x}_k$ is the (n X 1) system **state vector** at time t_k

Φ_k is the (n X n) **transition matrix** which relates $\underline{x}_k$ to $\underline{x}_{k+1}$ in the absence of a forcing function

Γ_k is the (n X n) **process noise distribution matrix** which transforms the $\underline{w}_k$ vector into the coordinates of $\underline{x}_k$

$\underline{w}_k$ is a (n X 1) white **disturbance sequence** or **process noise sequence** with known covariance structure.

$\underline{z}_k$ is a (m X 1) **measurement** at time t_k

H_k is a (m X n) **measurement matrix** or **observation matrix** relating $\underline{x}_k$ to $\underline{z}_k$ in the absence of measurement noise

v_k is a (m X 1) white **measurement noise sequence** with known covariance structure

• The covariance matrices for the white sequences are:

$$E(\bar{w}_k \bar{w}_i^T) = \delta_{ik} Q_k$$

$$E(\bar{v}_k \bar{v}_i^T) = \delta_{ik} R_k$$

$$E(\bar{w}_k \bar{v}_i^T) = 0, \forall(i, k)$$

where δ_{ik} is the Kronecker delta.

• Hence, we assume that process and measurement noise sequences are uncorrelated in time (white) and uncorrelated with each other.

2.3 Transition Matrix

• In order to implement the Kalman filter, a continuous time system model must be converted to a discrete one. Generally, this is far from easy. However, for our purposes, it suffices to know that the time continuous matrix differential equation:

$$\dot{\underline{x}} = F\underline{x}$$

can always be transformed into:

$$\underline{x}_{k+1} = \Phi_k \underline{x}_k$$

The only question is how hard it is to do.

• When the F matrix is constant in time and the equation is linear (no elements of x occur inside F), then the transition matrix is a function only of the time step Δt and it is given by the matrix exponential:

$$\Phi_k = e^{F\Delta t} = I + F\Delta t + \frac{(F\Delta t)^2}{2!} + \ldots$$

• In practice, the transition matrix can often be written by inspection. When this is not possible, writing a few terms of the above series often generates recognizable series in each element of the matrix partial sum, and the general form for each term can be generated by inspection. Other times, higher powers of F conveniently vanish anyway.

• When Δt is much smaller than the dominant time constants in the system, just the two-term approximation:

$$\Phi_k = e^{F\Delta t} = I + F\Delta t$$

is sufficient.

2.4 Low Dynamics Assumption

• When F depends on time, so does Φ and it satisfies the matrix version of the same differential equation as the state vector thus:

$$\boxed{\frac{d\Phi}{dt} = F(t)\Phi}$$

• If F is assumed to be slowly varying relative to Δt, then the matrix exponential can still be used. This will be called the **low dynamics assumption**. It is a big assumption as the time step gets larger.

• Notice that the model presents how the measurements are derived from the state, that is, the operation of the sensor itself. Often, in other applications, the process which converts a measurement into a state estimate is considered. That is, the problem of ***perception***. However, this model is the simpler reverse process of ***sensing*** itself.

• This is important to keep in mind because the filter is able to use underdetermined measurements of state for this reason. For example, if a single range measurement is available, the filter can use it to attempt to estimate two position coordinates or even more. This situation cannot persist for too long a period of time but single underdetermined measurements of multidimensional state vectors are quite legal.

3 Linear Discrete Time Kalman Filter

• The state space Kalman filter propagates both the state and its covariance forward in time[1], given an initial estimate of the state.

3.1 Filter Equations

• The Kalman filter equations for the linear system model are as follows:

Kalman filter

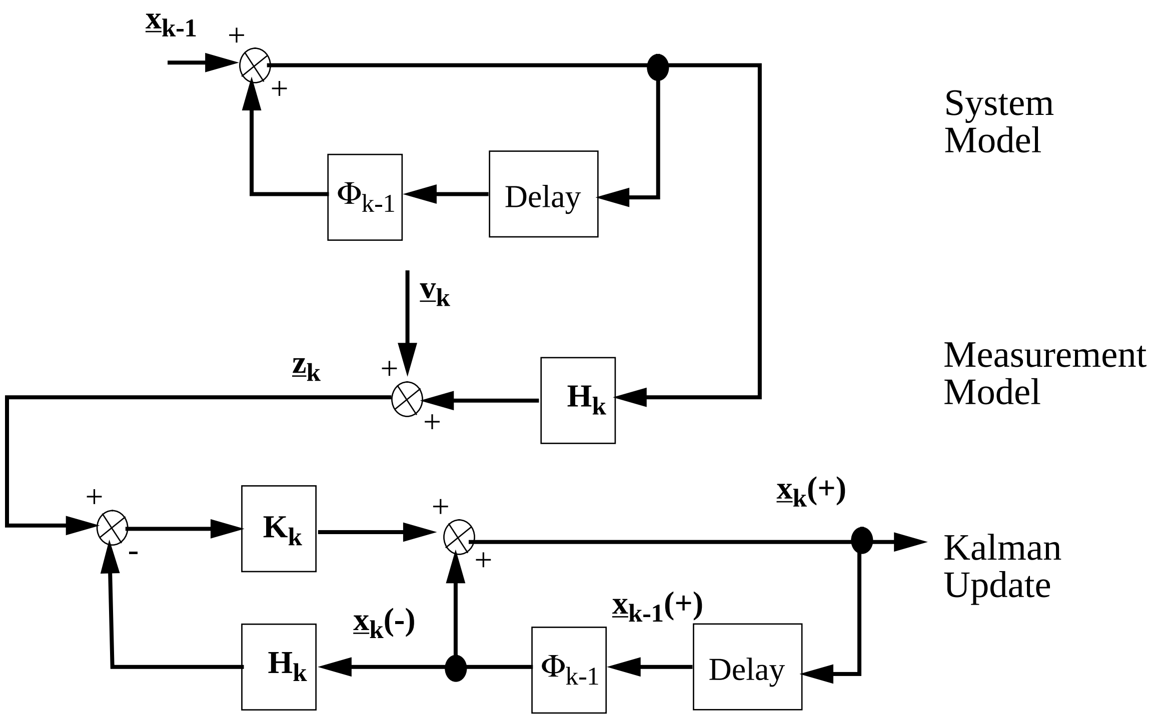
$$\begin{cases} K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1} & \text{compute Kalman gain} \\ \hat{x}_k = \hat{x}_k^- + K_k [z_k - H_k \hat{x}_k^-] & \text{update state estimate} \\ P_k = [I - K_k H_k] P_k^- & \text{update its covariance} \end{cases}$$

system model

$$\left.\begin{cases} \hat{x}_{k+1}^- = \Phi_k \hat{x}_k \\ P_{k+1}^- = \Phi_k P_k \Phi_k^T + \Gamma_k Q_k \Gamma_k^T \end{cases}\right\} \text{project ahead state and covariance}$$

1. The state estimate prior to incorporation of any new measurements will be denoted by $\hat{x}^-$, where the hat denotes an estimate and the super minus denotes the estimate prior to incorporation of the measurements (running one iteration of the filter equations).

• These can be visualized in block diagram form as follows:



• The equations obviously reduce to the last set of Kalman filter equations when the state vector is constant $\Phi_k = I$ and it has no temporal uncertainty growth associated with it $Q_k = 0$.

$$K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1}$$

$$\hat{x}_k = \hat{x}_k^- + K_k [z_k - H_k \hat{x}_k^-]$$

$$P_k = [I - K_k H_k] P_k^-$$

• ***The equations are not run all at once***. The last two run at high frequency and the first three are run when measurements are available.

• An advantage of this formulation is that it requires inversions of matrices of order m (number of measurements) which is usually less than n (the number of states). Indeed, it is possible to, under assumptions necessary for other reasons, set m to 1 and ***avoid matrix inversion completely***.

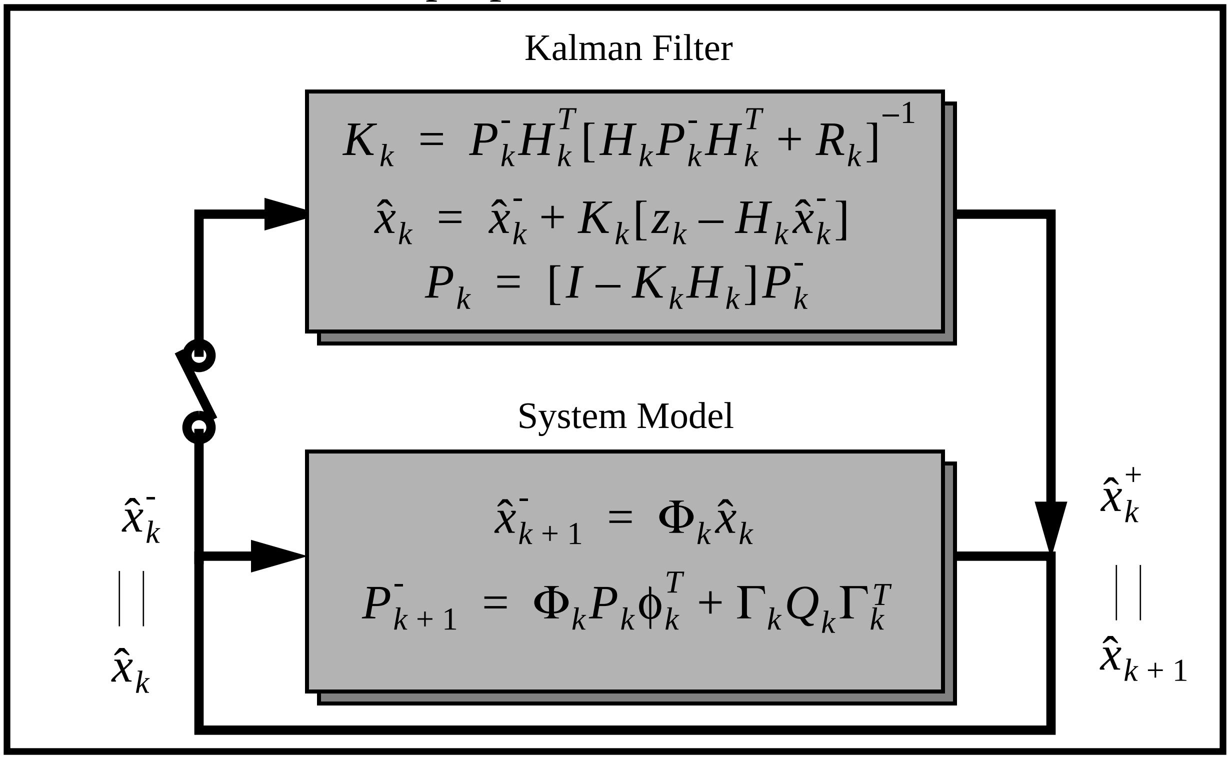
3.2 Time and Updates

• It is important to distinguish the ***time element*** from the ***arrival of measurements***.

- • The projection of the system state forward in time proceeds based ***solely on a measurement of time***.
- • Measurements are conceptualized as indirect measurements of state, and they arrive intermittently. When they arrive, they are incorporated into the state estimate through the Kalman gain but they are not strictly necessary.

• The number of measurements m, may be greater or less than n, and they may be redundant measurements of the same quantity.

• Whenever a measurement is available, the switch closes ***after the state has been predicted for that cycle*** by the system model, and the filter proper is executed.



3.3 Uncertainty Transformation

• If $\bar{z}$ is an arbitrary measurement of the vector $\bar{x}$ through some nonlinear relationship:

$$\bar{z} = f(\bar{x})$$

• Then it can be easily shown using a Taylor series approximation and the definition of the covariance matrix that:

$$Cov(\Delta z) = Exp[\Delta z \Delta z^T] = HExp[\Delta x \Delta x^T]H^T = HCov(\Delta x)H^T$$

• where H is the Jacobian of f. This relationship is responsible for the terms involving the measurement matrix H, the transition matrix Φ, and the Γ matrix in the Kalman filter equations.

3.4 Sequential Measurement Processing

• It can be shown that processing uncorrelated measurements one at a time gives the same result as processing them as one large block.

• This has extreme advantages in real-time systems with intermittent asynchronous sensor suites. It allows fairly modular software implementations which adapt in real time to the presence or absence of measurements at any particular time step.

• Thus, the software complication involved in restructuring the matrices to accommodate presence or absence of measurements can be completely avoided.

• This technique has computational advantages as well since inverting two matrices of order n/2 is much cheaper than inverting one of order n.

3.5 The Uncertainty Matrices

• It is important to distinguish the different roles of the three covariance matrices in the equations.

• The Q_k matrix models the uncertainty which corrupts the system model.

• The R_k matrix models the uncertainty associated with the measurement. For example, the element to be entered into R_k for a potentiometer is the number of counts of noise on the pot output.

• Finally, the P_k matrix gives the total uncertainty of the state estimate as a function of time and is managed by the filter itself.

4 Observability

• Situations may arise where there are not enough measurements in the entire sensor suite to predict the system state over the long term. These are called **observability** problems and they can be detected when diagonal elements of P_k are diverging with time.

• Observability problems can be fixed by reducing the number of state variables (i.e by incorporating the assumption that some are not too relevant) or by adding additional sensors. Observability is a property of the entire model including both the system and the measurement model, so it changes every time the sensors change.

• Formally, ***a system is observable if the initial state can be determined by observing the output for some finite period of time***. Generalizing from Gelb, consider the discrete, nth order, constant coefficient linear system:

$$x_{k+1} = \Phi x_k$$

for which there are m noise free measurements:

$$z_k = H x_k \quad k = 0, m-1$$

where each H is an m X n matrix. The sequence of the first n measurements can be written as:

$$\begin{aligned}
z_0 &= H x_0 \\
z_1 &= H x_1 = H\Phi x_0 \\
z_2 &= H x_2 = H(\Phi)^2 x_0 \\
&\ \ \cdots \\
z_{n-1} &= H x_{n-1} = H(\Phi)^{n-1} x_0
\end{aligned}$$

• This can be rewritten as the augmented set of equations:

$$Z = \Xi^T x_0 \qquad \text{or} \qquad Z^T = x_0^T \Xi$$

• If the initial state x_0 is to be determined from this sequence of measurements then, the matrix:

$$\Xi = \left[H^T \middle| \Phi^T H^T \middle| \dots \middle| (\Phi^T)^{n-1} H^T \right]$$

must have rank[1] n.

1. Recall that the rank of a matrix is the size of the largest nonzero determinant that can be formed from it. The rank of an m X n matrix can be no larger than the smaller of m and n. A square n X n matrix of rank n is called ***nonsingular***. The rank of the product of matrices is never larger than the smallest rank of the matrices forming the product.

5 Linearization

• The filter formulation presented earlier is based on a linear systems model and it is therefore not applicable in situations when either the system model or the measurement relationships are nonlinear. Consider an exact nonlinear model of a system as follows:

$$\dot{\underline{x}} = f(\underline{x}, t) + g(t)\underline{w}(t)$$
$$\underline{z} = h(\underline{x}, t) + \underline{v}(t)$$

Where f and h are vector nonlinear functions and $\underline{w}$ and $\underline{v}$ are white noise processes with zero crosscorrelation. Let the actual trajectory of the system be written in terms of an approximate trajectory $\bar{x}^*(t)$ and an error trajectory $\Delta\bar{x}(t)$ as follows:

$$\bar{x}(t) = \bar{x}^*(t) + \Delta\bar{x}(t)$$

5.1 Linearized Models

• By substituting this back into the model and approximating f and h by their Jacobians evaluated at the reference trajectory:

$$\Delta\dot{\bar{x}} = \frac{\partial f}{\partial \bar{x}}(\bar{x}^*, t)\Delta\bar{x} + g(t)\bar{w}(t)$$
$$\bar{z} - h(\bar{x}^*, t) = \frac{\partial h}{\partial \bar{x}}(\bar{x}^*, t)\Delta\bar{x} + \bar{v}(t)$$

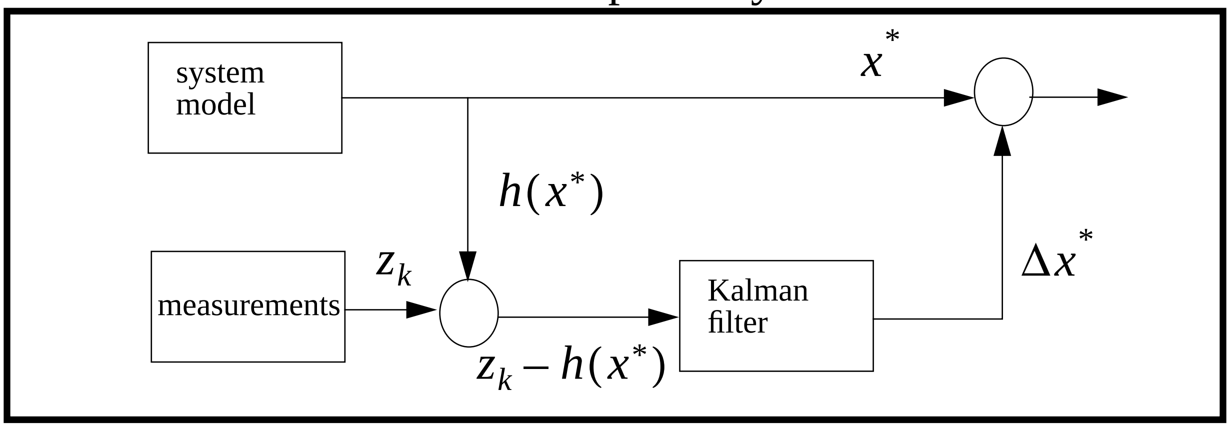
• Two different forms of Kalman filter can be generated from this linear assumption. The precise distinction between the two forms of filter is based on the measurement function $h(x^*)$, that is, whether it is updated based on the ***corrected*** (extended filter) or the ***nominal*** (linearized filter) trajectory.

5.2 Linearized Kalman Filter

• This **linearized system model** can be used to implement a **linearized Kalman filter** because the error dynamics and error measurement relationships are linear. In this model:

- • The deviation from the reference trajectory is the state vector.
- • The measurements are the true measurements less that predicted by the nominal trajectory.

• The linearized filter is used in a **feedforward** configuration as shown below where the nominal trajectory is ***not*** updated to reflect the error estimates computed by the filter:



• An advantage of the feedforward approach is that the unfiltered system model output provides high-fidelity response in the presence of high dynamics.

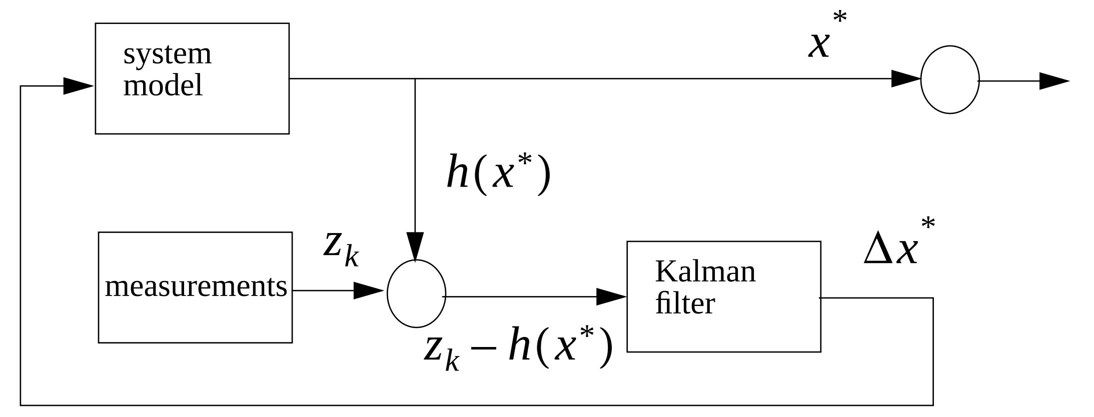
• However, such a filter is difficult to use for extended missions because, the reference trajectory eventually diverges to the point where the linear assumption is no longer valid across the variation in the state vector.

• The feedforward model can be used to integrate an INS with other navigation aids by considering its output to be the reference trajectory.

6 Extended Discrete Time Kalman Filter

• In the **extended Kalman filter**, the trajectory error estimates are used to update the reference trajectory as time evolves. This has the advantage that it is more applicable to extended missions.

• The extended filter can be visualized in a **feedback** configuration as shown below:



• In the case of the extended Kalman filter, it is possible to formulate the filter in terms of the state variables themselves rather than the error states.

• The discrete time extended Kalman filter equations for the system model are now as follows:

$\dot{\underline{x}} = f(\underline{x}, t) + g(t)\underline{w}(t)$	system model
$\underline{z} = h(\underline{x}, t) + \underline{v}(t)$	measurement model

$Q_k = E(\bar{w}_k \bar{w}_k^T)$	process noise
$R_k = E(\bar{v}_k \bar{v}_k^T)$	measurement noise

Kalman filter

$H_k = \frac{\partial h}{\partial \bar{x}}(\hat{x}_k^-)$	compute measurement Jacobian
$F_k = \frac{\partial f}{\partial \bar{x}}(\hat{x}_k^-)$	compute system Jacobian
$K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1}$	compute Kalman gain
$\hat{x}_k = \hat{x}_k^- + K_k[z_k - h(\hat{x}_k^-)]$	update state estimate
$P_k = [I - K_k H_k] P_k^-$	update its covariance

system model

$\hat{x}_{k+1}^- = \phi_k(\hat{x}_k)$	project state ahead
$P_{k+1}^- = \Phi_F P_k \Phi_F^T + \Gamma_k Q_k \Gamma_k^T$	update its covariance

where the usual conversion to the discrete time model has been performed.

6.1 State Transition for Nonlinear Problems

• When the system differential equation is nonlinear:

$$\dot{\underline{x}} = f(\underline{x}(t), t)$$

the state propagation equation will be written as:

$$\hat{x}_{k+1}^{-} = \phi_k(\hat{x}_k)$$

• the "transition matrix" can be generated from time linearization:

$$\underline{x}_{k+1} = \underline{x}_k + \dot{\underline{x}}_k \Delta t$$
$$\underline{x}_{k+1} = \underline{x}_k + f(\underline{x}_k, t_k)\Delta t$$

• The transition matrix can be written by inspection as an identity matrix with linear and angular velocity cross terms $f(\underline{x}_k, t_k)$ added and multiplied by Δt.

6.2 System Jacobian for Nonlinear Problems

• The system Jacobian for the EKF comes from state space linearization:

$$F = \frac{\partial}{\partial \bar{x}} f(\bar{x}, t)$$

6.3 Uncertainty Propagation for Nonlinear Problems

• In the nonlinear case, the uncertainty propagation equation is:

$$P_{k+1}^{-} = \Phi_F P_k \Phi_F^T + \Gamma_k Q_k \Gamma_k^T$$

• The matrix Φ_F is the transition matrix associated with the system Jacobian. The following eexpression for it is correct to first order and may be used in implementation:

$$\Phi_F = (I + F)\Delta t$$

6.4 The Measurement Conceptualization in the EKF

• It is important to recognize that the **measurement process itself** is used in the **extended** Kalman filter, and the computation of the deviation from predicted to actual measurement is automatic in the formalism. More specifically, the state update equation is:

$$\hat{x}_k = \hat{x}_k^- + K_k[z_k - h(\hat{x}_k^-)] \qquad \text{update state estimate}$$

• This contains the computation of the predicted measurement $h(\hat{x}_k^-)$ already. The predicted measurement is computed inside the filter itself.

7 State Vector Augmentation

• One of the secrets to high performance navigation systems is the mechanism by which they utilize the Kalman filter to **identify** themselves in real time.

• In the language of estimation theory, the mechanism is known as **state vector augmentation**.

• Both unknown system parameters and nonwhite noise sources can be modelled as the result of passing a white noise process through a system with linear dynamics.

7.1 Principle

• Suppose the measurement noise $\underline{v}$ in the continuous time model below is correlated:

$$\dot{\underline{x}} = F\underline{x} + G\underline{w}$$
$$\underline{z} = H\underline{x} + \underline{v}$$

• Oftentimes it is possible to consider that the correlated measurement noise arises through passing uncorrelated white noise $\bar{w}_1$ through a system with linear dynamics (i.e by filtering it) thus:

$$\dot{\underline{v}} = E\underline{v} + \underline{w}_1$$

• Using this model, the correlated component of noise can be considered to constitute a random element in the state vector. This element is added to the existing states to form the **augmented state vector**:

$$\frac{d}{dt}\begin{bmatrix} \underline{x} \\ \underline{v} \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & E \end{bmatrix}\begin{bmatrix} \underline{x} \\ \underline{v} \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \underline{w} \\ \underline{w}_1 \end{bmatrix}$$

• Then the measurement equation becomes:

$$\bar{z} = \begin{bmatrix} H & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{v} \end{bmatrix}$$

7.2 Parameter Identification

• Unknown system parameters can be modelled as deterministic new state variables with this technique.

• The (scalar) state differential equation for one new constant state is:

$$\dot{x}_i = 0$$

• The measurement matrix H is updated to reflect the new state.

• The basic operation of the filter is to project the measurement residual onto the states, so it will determine a value for this constant and even allow it to vary slowly with time.

8 Example - AHRS Dead Reckoning

• The filter equations are basically always the same because the last set of EKF formulas for nonlinear dynamic systems is extremely general. They can used for tracking an object in a viewfinder or estimating the position of a robot or for any problem we've seen so far in these notes.

• The hard issue for building a filter is the **modelling decisions** - the filling in of the matrices - just like in a controller design problem.

• This example is a very general dead reckoning 3D Kalman filter formulated for a redundant asynchronous sensor suite. This permits many measurement models to be expressed in closed form as scalar equations, which reduces the matrix computations to a minimum.

8.1 Design Decisions and Assumptions

• A few key assumptions permit the filter to perform as required:

• **Low Dynamics Assumption**. Linear and angular velocity are mostly constant between measurements. Makes it possible to reduce the state vector dimensionality to a minimum.

• **Taylor Remainder Theorem**. Under this basic theorem of calculus, provided the low dynamics assumption holds, uncertainty models for sensors and states can be generated from the last neglected term in the relevant Taylor series.

• **Principle Motion Assumption.** The vehicle moves primarily along the body y direction and rotates primarily about body z. This permits the filter to be formulated in observable form without the need for any landmark damping. It also eliminates two linear and two angular velocities from the state vector.

8.2 Why use a Kalman Filter

- **AHRS** is an acronym for Attitude and Heading Reference System. It measures roll, pitch, and yaw.
- Our dead reckoning system will also include wheel encoders, or a transmission encoder, and/or a Doppler groundspeed radar sensor, and a steering wheel encoder.
- Such a simple example coud be done without a Kalman filter but here are some advantages of using a Kalman filter:
    - **Redundant measurements**. Encoders and radar are redundant measurements of groundspeed. The filter will maximize the advantage of this redundancy - emphasizing one senor when the other is likely to be poor. For example, radar has a deadband..
    - **Integration of Dead Reckoning and Triangulation**. If any fix information is available, the formulation provides a simple mechanism for improving a dead reckoned estimate considerably.
    - **Modelling Frequency Response**. AHRS sensors tend to have long settling times. State vector augmentation can be used to model the frequency response of such sensors.
    - **Computational Inertial Force Compensation**. Measurements of path curvature and linear velocity can be used to compensate for centrifugal accelerations by removing them computationally from the clinometer outputs.

9 Example - System Model

• Most of the measurement and system model is a 3D kinematic model. All you need is kinematics - no dynamics.

• More generally, you have a system model when you have the details of the following equations:

$$\dot{\underline{x}} = f(\underline{x}, t) \qquad \text{system dynamics model}$$

$$\hat{x}_{k+1}^{-} = \phi_k(\hat{x}_k) \qquad \text{state transition}$$

$$F_k = \frac{\partial f}{\partial \underline{x}}(\hat{x}_k^{-}) \qquad \text{system jacobian}$$

9.1 State Vector

• The choice of state vector in this case will depend on the observability issue and it often does.

• Let x, y, z, θ, ϕ, and ψ denote the vehicle position and attitude in the navigation frame. If we use the state vector:

$$\underline{x} = \begin{bmatrix} x & y & z & \theta & \phi & \psi \end{bmatrix}^T$$

it turns out that the filter is not observable because vertical and side velocity - among other things - cannot be observed from measurements of forward velocity only.

• The solution is to reformulate things to ***explicitly*** assume that:

- the vehicle translates only along the body y axis
- the vehicle rotates only around the body z axis

• In this model, the state variables are:

$$\underline{x} = \begin{bmatrix} x & y & z & V & \theta & \phi & \psi & \dot{\beta} \end{bmatrix}^T$$

where V is the projection of the vehicle velocity onto the body y axis, and $\dot{\beta}$ is the projection of the vehicle angular velocity onto the body z axis.

9.2 System Dynamics Model

• Basically, the system model will be of the form:

$$V = \dot{\beta} = 0$$

but we also need to account for the state variable interrelationships.

• Using earlier results for the transformation of angular velocity, the continuous time system differential equations are as follows:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ V \\ \theta \\ \phi \\ \psi \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -Vs\psi c\theta \\ Vc\psi c\theta \\ Vs\theta \\ 0 \\ \dot{\beta}s\phi \\ -\dot{\beta}t\theta c\phi \\ \dot{\beta}c\phi/c\theta \\ 0 \end{bmatrix}$$

• This system model is ***nonlinear***, therefore it must be linearized according to the rules for an EKF.

9.3 System Jacobian

• The linearized continuous-time differential equation is:

$$\frac{d}{dt}\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\\ \Delta V\\ \Delta\theta\\ \Delta\phi\\ \Delta\psi\\ \Delta\dot{\beta}\end{bmatrix} = \begin{bmatrix}0&0&0&-s\psi c\theta&Vs\psi s\theta&0&-Vc\psi c\theta&0\\ 0&0&0&c\psi c\theta&-Vc\psi s\theta&0&-Vs\psi c\theta&0\\ 0&0&0&s\theta&0&0&Vc\theta&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\dot{\beta}c\phi&0&s\phi\\ 0&0&0&0&-\dot{\beta}c\phi/c^2\theta&\dot{\beta}t\theta s\phi&0&-t\theta c\phi\\ 0&0&0&0&\dot{\beta}s\theta c\phi/c^2\theta&-\dot{\beta}s\phi/c\theta&0&c\phi/c\theta\\ 0&0&0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\\ \Delta V\\ \Delta\theta\\ \Delta\phi\\ \Delta\psi\\ \Delta\dot{\beta}\end{bmatrix}$$

which gives the F matrix of the EKF.

9.4 Transition Matrix

• The transition matrix Φ does not exist for nonlinear plants, but the system differential equation can be approximated by linearizing in time as follows:

$$\begin{bmatrix}x\\ y\\ z\\ V\\ \theta\\ \phi\\ \psi\\ \dot{\beta}\end{bmatrix}_{K+1} = \begin{bmatrix}1&0&0&-s\psi c\theta dt&0&0&0&0\\ 0&1&0&c\psi c\theta dt&0&0&0&0\\ 0&0&1&s\theta dt&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&s\phi dt\\ 0&0&0&0&0&1&0&-t\theta c\phi dt\\ 0&0&0&0&0&0&1&c\phi dt/c\theta\\ 0&0&0&0&0&0&0&1\end{bmatrix}_K\begin{bmatrix}x\\ y\\ z\\ V\\ \theta\\ \phi\\ \psi\\ \dot{\beta}\end{bmatrix}_K$$

• This transition matrix is just the equations of 3D dead reckoning. It has been generated by re-expressing the nonlinear plant as a matrix function of the states, so it only appears to be linear.

10 Example - Measurement Models

• This section provides the 3D measurement models for such a filter incorporating many of the sensors commonly used on autonomous vehicles.

• One of the advantages of the body frame system model is that almost all of the measurement models are trivial. The tradeoff is that the system model has rotation transforms in it.

• More generally, you have a measurement model when you have the details of the following equations:

$$\underline{z} = h(\underline{x}, t) \qquad \text{measurement model}$$

$$H_k = \frac{\partial h}{\partial \bar{x}}(\hat{x}_k^-) \qquad \text{measurement Jacobian}$$

• In the filter implementation, it will be necessary to generate the value of the **predictive measurement** $h(\underline{x}, t)$ the measurement that is predicted from the current state estimate. From it, we can then generate the measurement residual - which is the quantity multiplied by the Kalman gain:

$$z_k - h(\hat{x}_k^-)$$

10.1 Transmission Encoder

• A transmission encoder measures differential distance, so it can be considered to be a device which measures velocity.

$$z_{enc} = V \qquad H_{enc} = \frac{\partial z_{enc}}{\partial \bar{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10.2 Doppler Groundspeed Radar

• The Doppler sensor has a model similar to the transmission encoder.

$$z_{dop} = V \qquad H_{dop} = \frac{\partial z_{dop}}{\partial \bar{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10.3 AHRS

• The clinometers, compass, and gyros in an AHRS can be assumed to measure the vehicle attitude directly so their measurement matrices are the identity matrix.

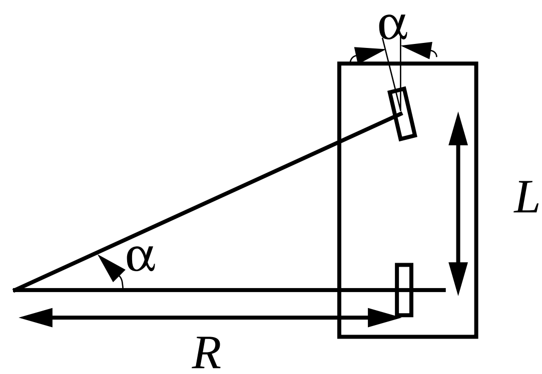
$$\underline{z}_{ahrs} = [I]\begin{bmatrix} \theta \\ \phi \\ \psi \end{bmatrix} \qquad H_{\rho} = \frac{\partial \underline{z}_{ahrs}}{\partial \underline{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

10.4 Steering Wheel Encoder

- A steering wheel encoder provides a measurement of the rate of rotation of the vehicle about the body z axis. Using the bicycle model approximation, the path curvature κ, radius of curvature R, and steer angle α are related by the wheelbase L.

$$\kappa = \frac{1}{R} = \frac{\tan\alpha}{L} = \frac{d\beta}{ds}$$

$$\dot{\beta} = \frac{d\beta}{ds}\frac{ds}{dt} = \kappa V = \frac{V\tan\alpha}{L}$$



- The steer angle α is the quantity indicated by the encoder. It is an indirect measurement of the ratio of $\dot{\beta}$ to velocity through the nonlinear measurement function:

$$\alpha = \operatorname{atan}\left(\frac{L\dot{\beta}}{V}\right) = \operatorname{atan}(\kappa L)$$

- The measurement Jacobian contains partial derivatives that will apparently approach infinity. If we are careful to remove apparent singularities we get:

$$H_\alpha = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial\alpha}{\partial V} & 0 & 0 & 0 & \frac{\partial\alpha}{\partial\dot{\beta}} \end{bmatrix}$$

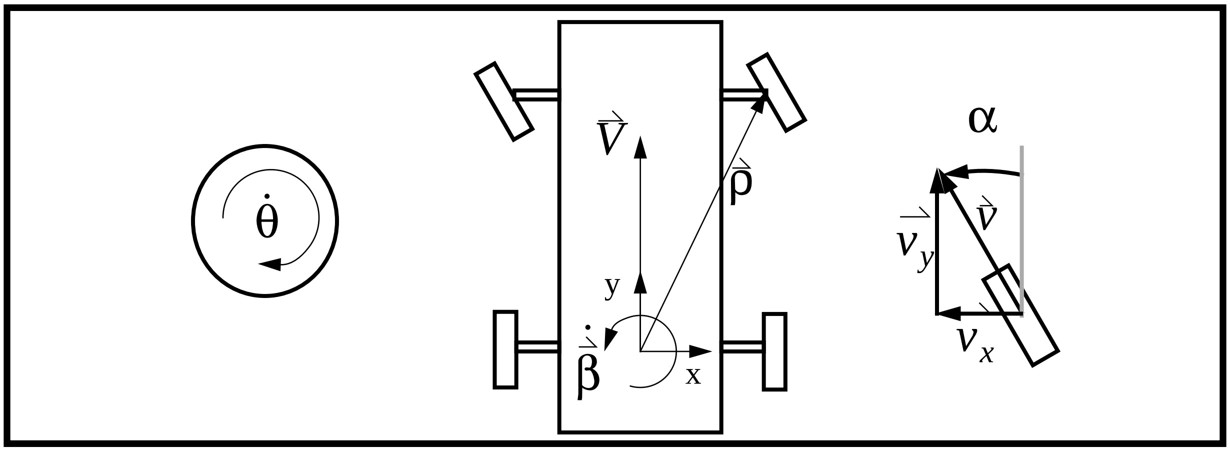
$$\frac{\partial\alpha}{\partial V} = \frac{-L\dot{\beta}}{1 + (L\dot{\beta})^2}$$

$$\frac{\partial\alpha}{\partial\dot{\beta}} = \left(\frac{LV}{1 + (L\dot{\beta})^2}\right)$$

- Physically, the steer angle is irrelevant when the vehicle is not moving, so the measurement must be discarded.

10.5 Wheel Encoders

- It is important to distinguish several kinds of measurements that may be available from instrumented wheels.
    - A **fixed wheel** is permitted to rotate about a single axis - the one associated with forward motion.
    - A **free wheel** may rotate about a vertical axis as well as the axis associated with forward motion.
    - Either of these two degrees of freedom may be powered or not and either may be instrumented or not.
- Consider a single wheel on a vehicle that has two degrees of rotational freedom as shown below. Let $\vec{\rho}$ be the position vector of the wheel relative to the vehicle control point. Let the wheel radius be r.



- It is simplest to ***formulate the measurements in the body frame***. The velocity of the end of the wheel axle relative to the world is available from vector algebra as:

$$\vec{v} = \vec{V} + \dot{\vec{\beta}} \times \vec{\rho} = V\hat{j} + \dot{\beta}\hat{k} \times (\rho_x \hat{i} + \rho_y \hat{j})$$

$$\vec{v}_x = -\dot{\beta}\rho_y \hat{i} \qquad \vec{v}_y = (V + \dot{\beta}\rho_x)\hat{j}$$

10.5.1 Steer Angle

• Now the available measurements actually invert these relationships. First the steer angle α of the wheel and its gradient are:

$$\alpha = \operatorname{atan}(\sigma) = \operatorname{atan}(v_y / v_x)$$

$$\frac{\partial\alpha}{\partial V} = \frac{\partial\alpha}{\partial\sigma}\frac{\partial\sigma}{\partial V} = \left(\frac{1}{1+\sigma^2}\right)\frac{1}{v_x} = -\left(\frac{1}{\dot{\beta}\rho_y}\right)\left(\frac{1}{1+\sigma^2}\right)$$

$$\frac{\partial\alpha}{\partial\dot{\beta}} = \frac{\partial\alpha}{\partial\sigma}\frac{\partial\sigma}{\partial\dot{\beta}} = \left(\frac{1}{1+\sigma^2}\right)\left(\frac{\rho_y v_y + \rho_x v_x}{v_x^2}\right) = (V\rho_y)\left(\frac{1}{1+\sigma^2}\right)$$

• This is a measurement of the ratio of angular to linear velocity and hence is a measure of curvature just as is the Ackerman steer angle.

10.5.2 Free Wheel Velocity

• A free wheel will rotate automatically about the body z axis by the necessary steer angle due to friction. Its measurement relationship in radians is:

$$\dot{\theta} = \frac{1}{r}\left(\sqrt{v_x^2 + v_y^2}\right) = \frac{1}{r}\left[\sqrt{(\dot{\beta}\rho_y)^2 + (V + \dot{\beta}\rho_x)^2}\right]$$

$$\frac{\partial}{\partial V}\dot{\theta} = \frac{1}{r}\left(\frac{2v_y}{2v}\right) = \sin(\alpha) \quad \frac{\partial}{\partial\dot{\beta}}\dot{\theta} = \frac{1}{r}\left(\frac{2v_x\rho_y + 2v_y\rho_x}{2v}\right) = \frac{1}{r}(v_x\cos\alpha + v_y\sin\alpha)$$

• This is a measurement that responds to both the linear and angular velocity of the vehicle but they cannot be distinguished from a single measurement.The filter will automatically distinguish linear and angular velocity when two or more wheel velocities are measured.

10.5.3 Fixed Wheel Velocity

• A fixed wheel will not rotate automatically about the body z axis. Its measurement relationship in radians is:

$$\dot{\theta} = \vec{v} \bullet \hat{j} = v_y / r = \frac{1}{r}(V + \dot{\beta}\rho_x)$$

$$\frac{\partial}{\partial V}\dot{\theta} = \frac{1}{r} \qquad \frac{\partial}{\partial \dot{\beta}}\dot{\theta} = \frac{\rho_x}{r}$$

• Again, this is a measurement that responds to both the linear and angular velocity of the vehicle but they cannot be distinguished from a single measurement. The filter will automatically distinguish linear and angular velocity when two or more wheel velocities are measured.

11 Example - Uncertainty Model

• Often, there is no knowledge of correlation of error sources, and both the process noise covariance matrix Q and the measurement covariance matrix R are assumed to be diagonal.

$$Q = diag\left[\sigma_x^2\ \sigma_y^2\ \sigma_z^2\ \sigma_V^2\ \sigma_\theta^2\ \sigma_\phi^2\ \sigma_\psi^2\ \sigma_{\dot{\beta}}^2\right]$$

$$R = diag\left[\sigma_{enc}^2\ \sigma_{dop}^2\ \sigma_{com}^2\ \sigma_{pitch}^2\ \sigma_{roll}^2\ \sigma_{yaw}^2\ \sigma_{steer}^2\right]$$

• The state covariance matrix P will automatically evolve off diagonal terms as the filter runs.

11.1 State Uncertainty

• The state uncertainty model represents the disturbances which excite the linear system. It estimates how bad things can get when the system is run open loop (i.e with no sensors) for a given period of time.

• In the absence of any other information, a plausible approach is to estimate error as the Taylor remainder in the dead reckoning equations, because, after all, dead reckoning is a truncated Taylor series in time. The Q_k matrix can be assumed diagonal, and its elements set to the predicted magnitude of the truncated terms in the constant velocity model.

11.1.1 Process Noise Distribution Matrix

• The Γ matrix allows us to represent the uncertainties of the variables in the body frame where they can be determined by intuition. it automatically converts coordinates to the navigation frame of the state vector. Let the Γ matrix be given by:

$$\Gamma = \begin{bmatrix} \left[R_b^n\right] & 0 & \left[0\right] & 0 \\ 0 & 1 & 0 & 0 \\ \left[0\right] & 0 & \left[\Omega\right] & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Q = diag\left[\sigma_x^2\ \sigma_y^2\ \sigma_z^2\ \sigma_V^2\ \sigma_\theta^2\ \sigma_\phi^2\ \sigma_\psi^2\ \sigma_{\dot{\beta}}^2\right]$$

11.1.2 Linear Position States

• The translational uncertainty can be set to one-half the maximum acceleration times the square of the time step. This is the error expected when constant velocity is assumed. This gives:

$$\sigma_x = \sigma_y = \sigma_z = \frac{a_{max}(\Delta t)^2}{2}$$

11.1.3 Angular Position States

• For the angular position states, there are no measurements of velocity in the DR equations for all axes, so the truncation error is a velocity term. A very rough error estimate for these is:

$$\sigma_\theta = \sigma_\phi = \sigma_\psi = \Omega_{max}\Delta t$$

Where Ω_{max} is the estimated highest angular velocity of the vehicle body.

11.1.4 Linear Velocity States

• In the case of linear velocity, there are no acceleration states which propagate it forward in time via the transition matrix, so this state will not move if its uncertainty is set to zero. Again, using the remainder theorem:

$$\boxed{\sigma_V = a_{max}\Delta t}$$

11.1.5 Angular Velocity States

• The angular velocity state also has no acceleration states to propagate it forward in time. Using the remainder theorem:

$$\boxed{\sigma_{\dot{\beta}} = \alpha_{max}\Delta t}$$

11.2 Measurement Uncertainty

• The measurement uncertainties are far more critical to the filter operation, because, after all, the whole system is considered to fail if sensors are lost for only a few seconds and filter optimality is not an issue.

11.2.1 Encoder & Doppler

• The random error in the encoder can be extimated to have a magnitude proportional to the distance travelled. Similiar statements apply to the Doppler radar. The uncertainty model is:

$$\boxed{\begin{aligned} \sigma_{enc} &= SFE_{enc}V_{enc} \\ \sigma_{dop} &= SFE_{dop}V_{dop} \end{aligned}}$$

11.2.2 Attitude

• In the absence of information, the best that can be done is to assume some constant uncertainty.

• However, it is common to find that yaw uncertainty is worse than that of the other two angles, because the former is often based on the gravity vector, and the latter on the weak and unreliable local magnetic field so we can set:

$$\sigma_\theta = \sigma_\phi = \sigma_{ATT} \qquad \sigma_\psi = 2\sigma_{ATT}$$

11.2.3 Steering

• Steering wheel position can be measured with any number of simple transducers. It is typically a very low-fidelity measurement that does not benefit from overly precise error characterization. Let:

$$\sigma_{steer} = \text{constant}$$

12 Example - Simple Terrain Aiding

• The pure dead reckoning filter of the previous section is unlikely to achieve an accuracy which exceeds a few percent of the distance travelled. This is because of the essential integration of errors in the process of dead reckoning.

• Whatever the fidelity of the measurements used in practical dead reckoning, a fix is needed at regular intervals to damp the DR and the mechanism for doing this is the subject of this section.

• The Kalman filter is an ideal formalism for integration of dead reckoning and position fixes because fixes are simply additional measurements which can be folded into the equations in like manner to the DR measurements.

12.1 Fixes in the Navigation Frame

• The simplest form of position fix is a direct measurement of the vehicle position in the navigation frame. The **survey point** is the only such fix available because position indicating devices cannot usually be mounted at the center of the body frame.

• The vehicle is positioned at a point which has been presurveyed in the nav frame. Once the filter is told that this is the case, it can use its stored knowledge of the true coordinates of the survey point to generate the fix.

• The measurement matrix for survey points is trivial:

$$\underline{z} = \begin{bmatrix} x_{sp} & y_{sp} & z_{sp} \end{bmatrix}^T \qquad H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

12.2 Fixes in a Positioner Frame

• Sensors which can provide fixes on their own position include the GPS receiver and the inertial navigation system. The measurement matrix for such a sensor is relatively trivial:

$$\underline{z} = T_b^p \left[x_{gps} \; y_{gps} \; z_{gps} \right]^T \qquad H = T_b^p \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

12.3 Fixes in the Perception Sensor Frame

• Let the position of a landmark in the navigation frame be known to be:

$$\underline{r}_L^n = \left[x_L \; y_L \; z_L \right]^T$$

• Consider a generalized perception sensor which generates a 3D image of which most real sensors are special cases. The measurment model involves the transformation from the navigation to the sensor frame:

$$\underline{r}_L^s = \begin{bmatrix} x_L^s \\ y_L^s \\ z_L^s \end{bmatrix} = T_b^s T_n^b(\underline{x}) \begin{bmatrix} x_L \\ y_L \\ z_L \end{bmatrix} = T_b^s T_n^b(\underline{x}) \underline{r}_L^n$$

Where $T_n^b(\underline{x})$ is the nav frame to body frame homogeneous transform, and T_b^s is the body frame to sensor frame homogeneous transform.

• The measurement matrix for the transform is evaluated as the product of the constant body to sensor transform, the Jacobian tensor of the nav to body transform, and the landmark position vector.

$$H_L^s = \frac{\partial}{\partial \underline{x}}(\bar{r}_L^s) = T_b^s \frac{\partial}{\partial \underline{x}}(T_n^b(\underline{x}))\underline{r}_L^n$$

• Now to express the generated image, let a generalized nonlinear imaging function map a point in the sensor frame into image coordinates:

$$\underline{r}_L^i = f(\underline{r}_L^s) = f(T_b^s T_n^b(\underline{x})\underline{r}_L^n)$$

• where $\bar{r}_L^i$ is the triple (range, azimuth, elevation) for a rangefinder or the pair (row, column) for a video camera. The complete measurement Jacobian is then given by the chain **rule:**

$$H_L = \left(\frac{\partial \underline{r}_L^i}{\partial \underline{r}_L^s}\right)\left(\frac{\partial}{\partial \underline{x}}(\underline{r}_L^s)\right) = H_s^i H_L^s = H_s^i T_b^s \frac{\partial}{\partial \underline{x}}(T_n^b(\underline{x}))\underline{r}_L^n$$

This is the general case for any sensor.

• Recall that the measurement Jacobian provides the information necessary to project the residual onto the state vector. The measurement uncertainty itself arises in the R_k matrix. So the analysis thus far has nothing to do with the sensor itself. Rather, it answers the question of how an error in vehicle position relates to an error in the position of a landmark in the image for a perfect sensor.

• In order to use this formula for any particular sensor, the imaging Jacobian must be substituted for the particular sensor used.

• The matrix partial is a tensor. Let it be 4 X 4 X n. Its second index is matched with the row index of the landmark position vector to generate the 4 X n matrix:

$$H_{i,k} = \left[\frac{\partial T}{\partial \underline{x}}\right]_{i,j,k} \underline{r}_j \qquad \begin{array}{l} i=1,4 \\ j=1,4 \\ k=1,n \end{array}$$

• Notice that the fact that the filter is using the difference between the projected position of the landmark and the actual position of the landmark in the image is not explicit. The operation of the filter is such that it automatically computes what the differential change in the state vector has to be in order for the observed measurement to be made.

13 Bandwidth and Efficiency

• Typically, measurements cannot reasonably be packaged to arrive synchronously. For example, Doppler readings may not be available as frequently as encoder readings because they are already filtered. GPS measurements cannot be generated faster than 2 Hz whereas inertial systems can be 100 times faster than this.

13.1 Asynchronous Implementation

• The state equations can be run at about 100 Hz while measurements are incorporated at whatever rate they are generated by the sensors. The basic algorithm is given below:

```
State_Update() /* entered every cycle */
  {
  update state estimate for a time step of dt
  via the transition matrix(dt);
  if( Doppler measurement available)
    run Kalman() on Doppler;
  if( Encoder measurement available)
    run Kalman() on encoder;
  if( AHRS measurement available)
    run Kalman() on AHRS;
    if( Steering measurement available)
    run Kalman() on steering;
  }
```

13.2 Efficiency

- It is typical to find that most of the time is spent computing the uncertainty matrices, and the Kalman gain, and in inverting and multiplying matrices. The following two steps make it possible to run the filter at 100 Hz on a SPARC 1.

13.2.1 Kalman Gain

- The matrix Kalman gain equation for an EKF is:

$$K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1}$$

- Let a single measurement arrive for integration with the state estimate. Then, the R matrix is a scalar:

$$R = [r]$$

- Let the measurement project onto a single state whose index is s with a coefficient of unity. Then the H matrix is a row vector with a single unit element in the s'th position:

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The expression $P_k^- H_k^T$ is the s'th column of P_k^- and the expression $H_k P_k^- H_k^T$ is the (s,s) element of P_k^-. Define:

$$p = P_{ss}$$

- Finally, the Kalman gain is a column vector equal to a constant times the s'th column of P_k^-:

$$K = \left(\frac{1}{p + r}\right) P_{is} \; \forall i$$

13.2.2 Uncertainty Propagation

- The matrix uncertainty propagation equation for an EKF is:

$$P = [I - (KH)]P$$

- This can be computed many times faster as:

$$P = P - K(HP)$$

and this rewrite is valid regardless of the form of the measurement matrix H.

- Further simplification is possible in the case of a single scalar measurement. Let a single measurement arrive for integration with the state estimate. Again, let the measurement project onto a single state whose index is s with a coefficient of unity. Then the H matrix is a row vector with a single unit element in the s'th position:

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The expression HP is then the s'th row of P_k^-. Reusing the last result, the expression KHP is simply a constant times the outer product of the s'th column and the s'th row of P_k^-:

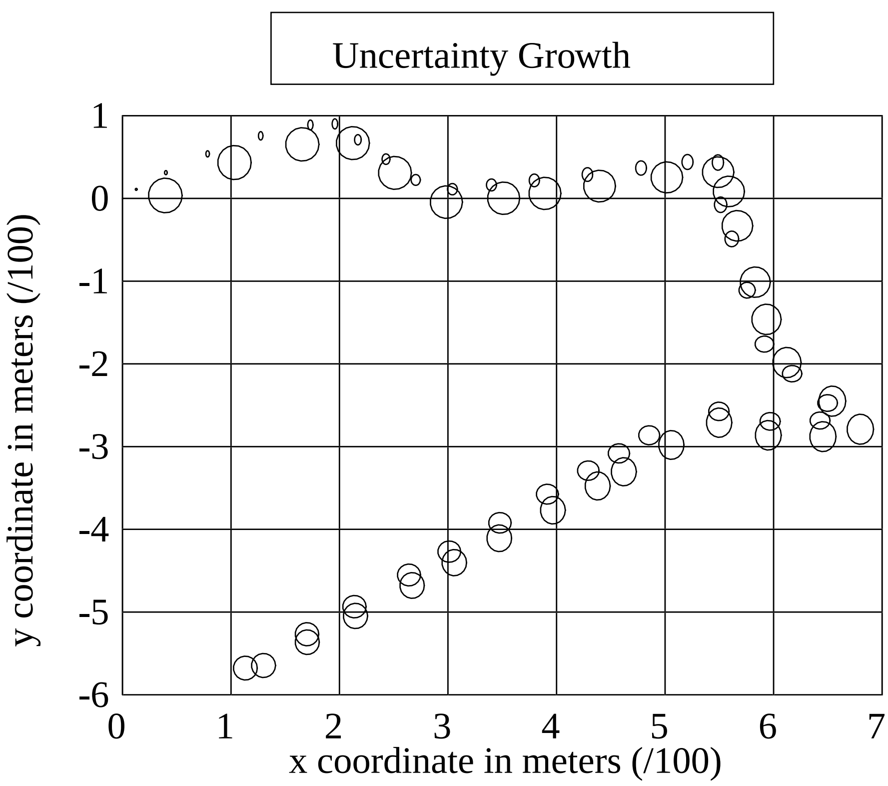
$$(KHP)_{ij} = \left(\frac{1}{p + r}\right) P_{is} P_{sj} \forall i \forall j$$

14 Results

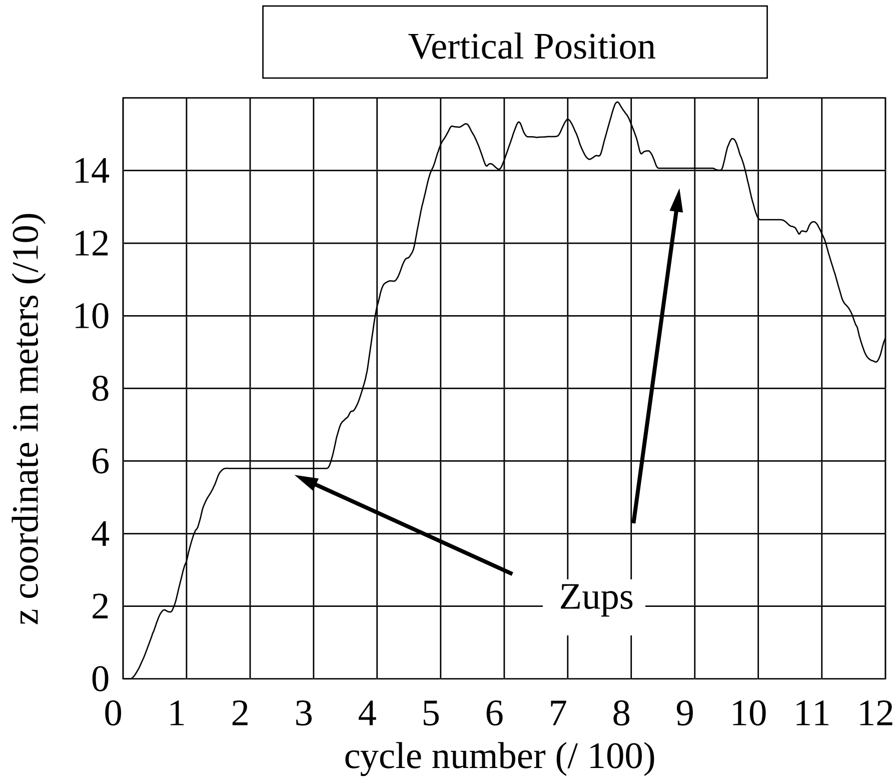
• This filter has been implemented and tested in the field. The sensors used were a steering wheel potentiometer for "yawrate" measurement, transmission encoder and redundant Doppler radar for measurement of velocity, and an AHRS for three-axis attitude.

• One of many test runs will be used to illustrate performance. In this run, winding mountainous city streets were driven. The total excursion was about 4 Km in the horizontal plane and 200 meters vertically.

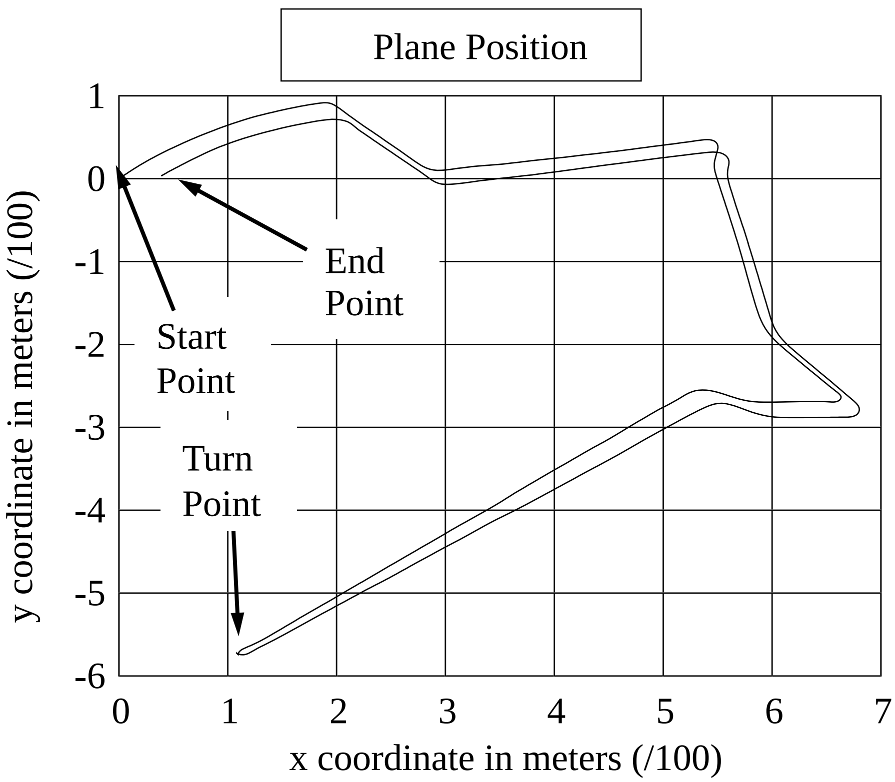
• The qualitatively correct growth of uncertainty is illustrated because the uncertainty ellipses touch the path when it was driven in the other direction. Point repeatability less than 1% of the travelled distance was normally achieved.



• One of the advantages of the 3D formulation is the availability of the z coordinate. A zup (zero velocity update) mode was included in order to check undesirable growth of the state uncertainty when the vehicle was stopped. Zups appear as flat regions because the abscissa is time.



• The position output in the plane is illustrated. Notice that the return path from the turn point could be rotated through a small angle at the turn point and the graphs would overlap almost perfectly. A residual systematic error in heading is responsible for this.



15 Summary

- The state space Kalman filter is the most generally useful form. It is really two sets of equations. The system model is run as fast as possible and the Kalman filter runs when measurements are available.
- When the system model or the measurement models are nonlinear, the Extended Kalman Filter is generated through a process of linearization.
- Observability is a concern that is intuitively obvious in simple cases.
- This form of filter intrinsically integrates odometric dead reckoning and landmark observations. The growth of compounded DR uncertainty, the GDOP of triangulation and the transformation of uncertainty are all handled automatically.
- A practical implementation deals with matters of asynchronous measurements and processing efficiency. the latter can be achieved by special case solutions for scalar direct measurements.