Extended Kalman Filter

If $\mathbf{h}_i(\mathbf{s}_i)$ is not linear or a linear relationship between $\mathbf{x}_i$ and $\mathbf{s}_i$ cannot be written down, the so-called *Extended Kalman Filter* (EKF for abbreviation) can be applied.

The EKF approach is to apply the standard Kalman filter (for *linear* systems) to *nonlinear* systems with additive white noise by continually updating a *linearization* around the previous state estimate, starting with an initial guess. In other words, we only consider a linear Taylor approximation of the system function at the previous state estimate and that of the observation function at the corresponding predicted position. This approach gives a simple and efficient algorithm to handle a nonlinear model. However, convergence to a reasonable estimate may *not* be obtained if the initial guess is poor or if the disturbances are so large that the linearization is inadequate to describe the system.

We expand $\mathbf{f}_i(\mathbf{x}'_i, \mathbf{s}_i)$ into a Taylor series about $(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1})$:

$$\begin{aligned} \mathbf{f}_i(\mathbf{x}'_i, \mathbf{s}_i) &= \mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1}) + \frac{\partial \mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1})}{\partial \mathbf{x}'_i}(\mathbf{x}'_i - \mathbf{x}_i) + \frac{\partial \mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1})}{\partial \mathbf{s}_i}(\mathbf{s}_i - \hat{\mathbf{s}}_{i|i-1}) \\ &\quad + O((\mathbf{x}'_i - \mathbf{x}_i)^2) + O((\mathbf{s}_i - \hat{\mathbf{s}}_{i|i-1})^2) \ . \end{aligned} \tag{22}$$

By ignoring the second order terms, we get a linearized measurement equation:

$$\mathbf{y}_i = M_i \mathbf{s}_i + \boldsymbol{\xi}_i \ , \tag{23}$$

where $\mathbf{y}_i$ is the new measurement vector, $\boldsymbol{\xi}_i$ is the noise vector of the new measurement, and M_i is the linearized transformation matrix. They are given by

$$\begin{aligned} M_i &= \frac{\partial \mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1})}{\partial \mathbf{s}_i} \ , \\ \mathbf{y}_i &= -\mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1}) + \frac{\partial \mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1})}{\partial \mathbf{s}_i}\hat{\mathbf{s}}_{i|i-1} \ , \\ \boldsymbol{\xi}_i &= \frac{\partial \mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1})}{\partial \mathbf{x}'_i}(\mathbf{x}'_i - \mathbf{x}_i) = -\frac{\partial \mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1})}{\partial \mathbf{x}'_i}\eta_i \ . \end{aligned}$$

Clearly, we have $E[\boldsymbol{\xi}_i] = \mathbf{0}$, and $E[\boldsymbol{\xi}_i \boldsymbol{\xi}_i^T] = \frac{\partial \mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1})}{\partial \mathbf{x}'_i} \Lambda_{i} \frac{\partial \mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1})}{\partial \mathbf{x}'_i}^T \triangleq \Lambda_{i}$.

The extended Kalman filter equations are given in the following algorithm, where the derivative $\frac{\partial \mathbf{h}_i}{\partial \mathbf{s}_i}$ is computed at $\mathbf{s}_i = \hat{\mathbf{s}}_{i-1}$.

Algorithm: Extended Kalman Filter

- Prediction of states:
$$\hat{\mathbf{s}}_{i|i-1} = \mathbf{h}_i(\hat{\mathbf{s}}_{i-1})$$
- Prediction of the covariance matrix of states:
$$P_{i|i-1} = \frac{\partial \mathbf{h}_i}{\partial \mathbf{s}_i} P_{i-1} \frac{\partial \mathbf{h}_i}{\partial \mathbf{s}_i}^T + Q_{i-1}$$
- Kalman gain matrix:
$$K_i = P_{i|i-1} M_i^T (M_i P_{i|i-1} M_i^T + \Lambda_{\ i})^{-1}$$
- Update of the state estimation:
$$\begin{aligned} \hat{\mathbf{s}}_i &= \hat{\mathbf{s}}_{i|i-1} + K_i(\mathbf{y}_i - M_i \hat{\mathbf{s}}_{i|i-1}) \\ &= \hat{\mathbf{s}}_{i|i-1} - K_i \mathbf{f}_i(\mathbf{x}_i, \hat{\mathbf{s}}_{i|i-1}) \end{aligned}$$
- Update of the covariance matrix of states:
$$P_i = (\mathbf{I} - K_i M_i) P_{i|i-1}$$
- Initialization:
$$P_{0|0} = \Lambda_{\mathbf{s}_0} \quad \text{and} \quad \hat{\mathbf{s}}_{0|0} = E[\mathbf{s}_0]$$

Zhengyou Zhang
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