Robotic Motion Planning: Probabilistic Primer

Robotics Institute 16-735

http://voronoi.sbp.ri.cmu.edu/~motion

Howie Choset

http://voronoi.sbp.ri.cmu.edu/~choset

Experiments and outcomes

- Experiment: Flip a Coin
- Outcome: Heads or Tails

- Experiment: Person’s temperature in class now
- Outcome: a scalar

- Event Subset of possible outcomes $E \subset S$

Probability

- Pr(*E*): Probability of an event *E* occurring when an experiment is conducted
- Pr maps *S* to unit interval

$$\Pr(\text{heads}) = 0.5, \Pr(\text{tails}) = 0.5$$

$$\Pr(\text{heads} \cup \text{tails}) = 1$$

1. $0 \leq \Pr(E) \leq 1$ for all $E \subset \mathcal{S}$.
2. $\Pr(\mathcal{S}) = 1$.
3. $\sum_i \Pr(E_i) = Pr(E_1 \cup E_2 \cup \ldots)$ for any countable disjoint collection of sets $E_1, E_2, \ldots$. This property is known as *sigma additivity*. In particular, we have $\sum_{i=1}^{n} \Pr(E_i) = Pr(E_1 \cup E_2 \cup \ldots \cup E_n)$.
4. $\Pr(\emptyset) = 0$.
5. $\Pr(E^c) = 1 - \Pr(E)$, where E^c denotes the complement of E in $\mathcal{S}$.
6. $\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2)$.

Dependence

- Independent: $\Pr(E_1 \cap E_2) = \Pr(E_1)\Pr(E_2)$
- Conditional Probability:
  - If E_1 and E_2 are independent, then $\Pr(E_1|E_2) = \Pr(\mathrm{E}_1)$
  - Bayes Rule

$$\Pr(E_1|E_2) = \frac{\Pr(E_2|E_1)\Pr(E_1)}{\Pr(E_2)}.$$

More Bayes Rule

$$p(a|b) = \frac{p(b|a)\ p(a)}{p(b)}$$

$$p(a|b,c) = \frac{p(b|a,c)\ p(a|c)}{p(b|c)}$$

Total Probability

Discrete

$$p(a) = \sum_i p(a \wedge b_i)$$
$$= \sum_i p(a \mid b_i)\, p(b_i)$$

Continuous

$$p(a) = \int p(a \mid b)\, p(b)\, db$$

it follows that:

$$p(a \mid b) = \int p(a \mid b, c)\, p(c \mid b)\, dc$$

Random Variable

- A mapping from events to a real number $X : \mathcal{S} \rightarrow \mathbb{R}$

- Examples
  - Discrete: heads or tails, number of heads for repeated flips
  - Continuous: temperature

- Random Vector: $X : \mathcal{S} \rightarrow \mathbb{R}^n$

Distribution

- Any statement one can make about a random variable

- Cumulative Distribution Function (CDF): $F_X(a) = \Pr(X \leq a)$

- Probability ______ Function (P_F):
  - Discrete: Mass (PMF) $f_X(a) = \Pr(X = a)$
  - Continuous: Density (PDF) $\Pr(a \leq X \leq b) = \int_{x=a}^{b} f_X(x)dx$

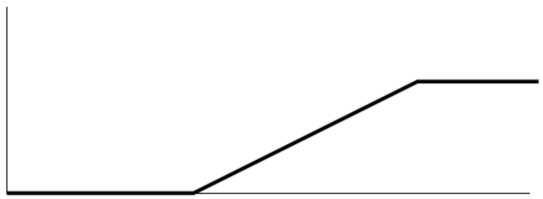
Note that $Pr((X = a)) = \int_{x=a}^{a} f_X(x)dx$ = 0

Uniform Distribution

CDF

$$U(x;a,b) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x \ge b \end{cases}$$

$$Pr(a' \le x \le b') = U(b';a,b) - U(a';a,b)$$



PDF

$$u(x;a,b) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \le x \le b \\ 0 & x \ge b. \end{cases}$$

$$Pr(a' \le x \le b') = \int_{x=a'}^{b'} u(x;a,b)dx$$

Expected Value

- PMF

$$E(X) = \sum_i x_i f_X(x_i)$$

PDF

$$E(X) = \int_{x \in \mathbb{R}^n} x f_X(x) dx$$

$$E(aX + bY) = aE(X) + bE(Y)$$

$$E(X) \text{ with } \bar{X}$$

Variance and Co-variance

- Variance: $\sigma^2 = E((X - \bar{X})^2)$

  Or σ_i^2 for X_i

- Co-Variance $\sigma_{ij} = E((X_i - \bar{X}_i)(X_j - \bar{X}_j))$

  $\sigma_{ij} = 0$, X_i and X_j are independent

- Co-Variance Matrix $P_X = E\left((X - \bar{X})(X - \bar{X})^T\right)$

  σ_i^2 Diagonal terms

  σ_{ij} Off-Diagonal Terms

Gaussians

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^n |P_X|}} e^{-\frac{1}{2}(x-\bar{X})^T P_X^{-1}(x-\bar{X})}$$

$P_X \in \mathbb{R}^{n \times n}$

$\bar{X} \in \mathbb{R}^n$ is the mean vector