15-887
Planning, Execution and Learning

A and Weighted A* Search*

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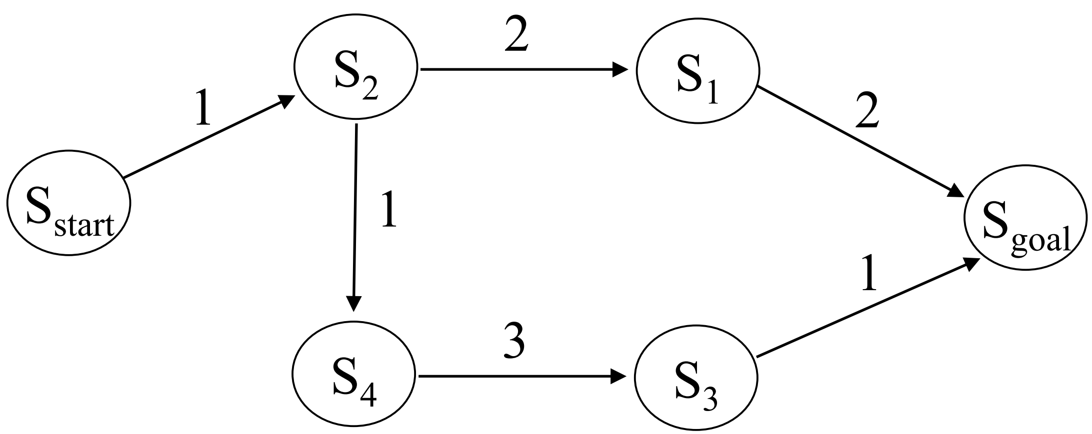
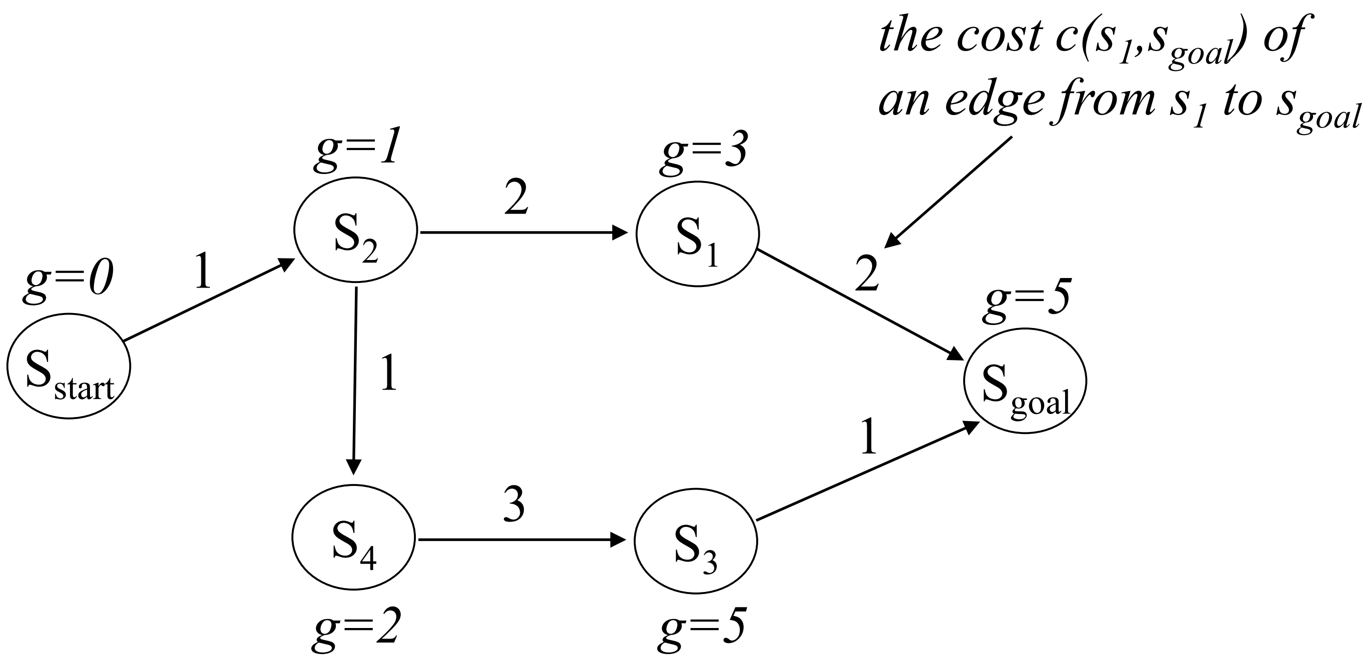
Searching Graphs for a Least-cost Path

- Once a graph is constructed (from skeletonization or uniform cell decomposition or adaptive cell decomposition or lattice or whatever else), we need to search it for a least-cost path



Searching Graphs for a Least-cost Path

- Many searches work by computing optimal g-values for relevant states
  - $g(s)$ – an estimate of the cost of a least-cost path from s_{start} to s
  - optimal values satisfy: $g(s) = \min_{s'' \in pred(s)} g(s'') + c(s'',s)$



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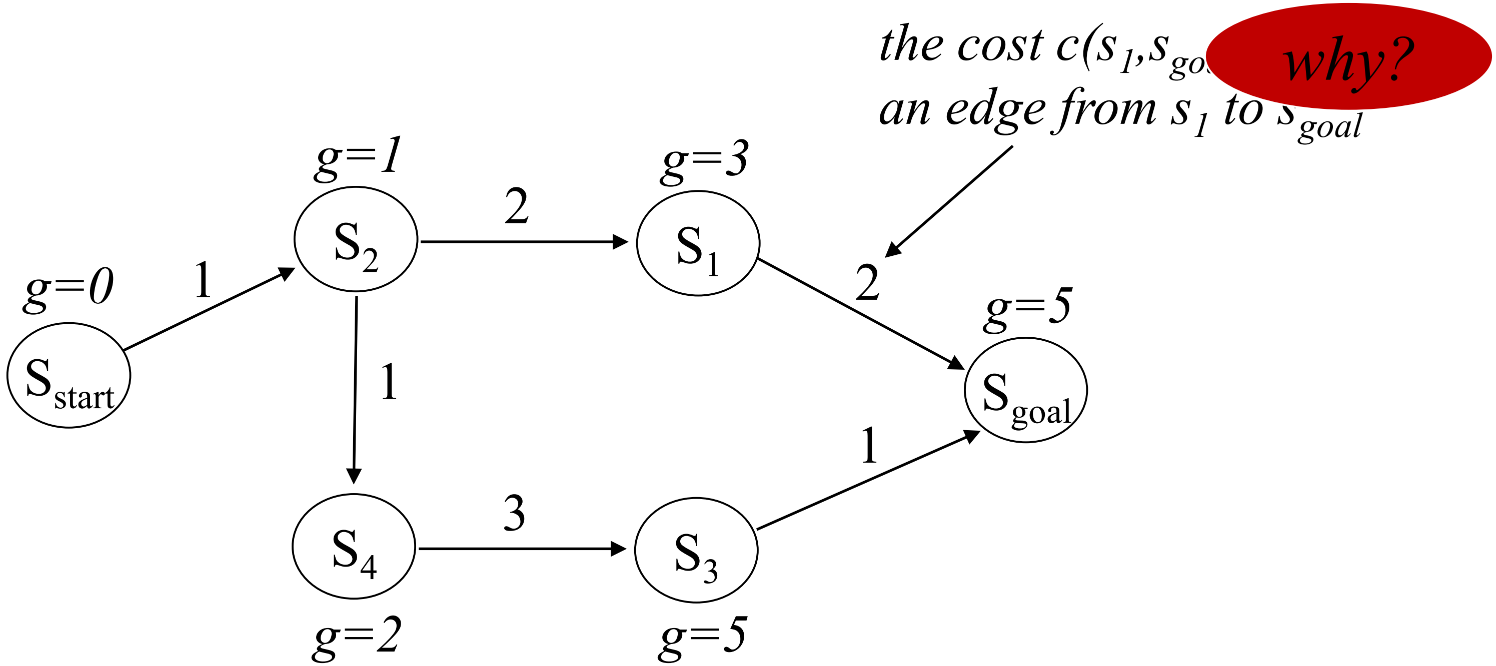
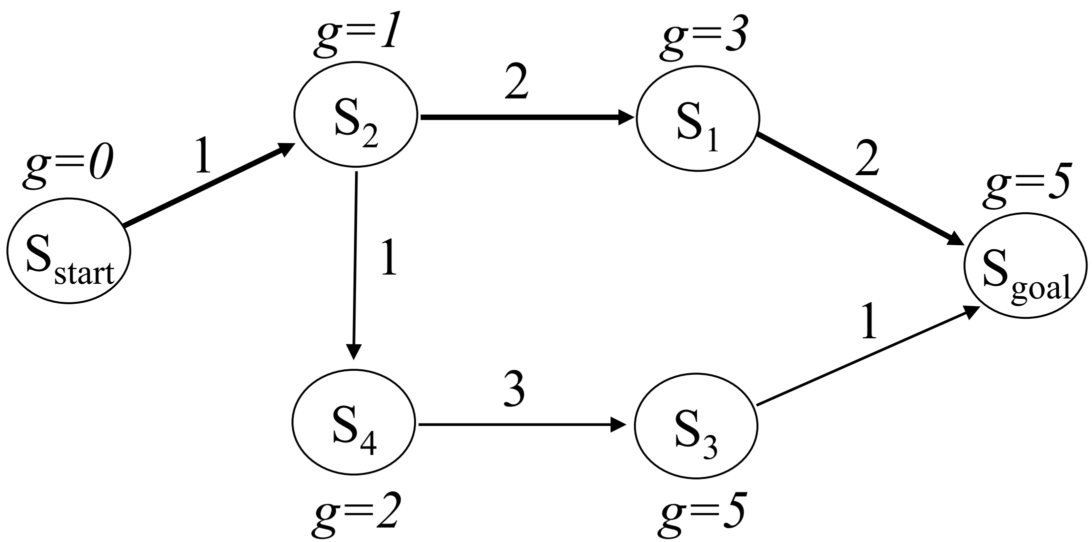


Searching Graphs for a Least-cost Path

- Least-cost path is a greedy path computed by backtracking:
  - start with s_{goal} and from any state s move to the predecessor state s' such that

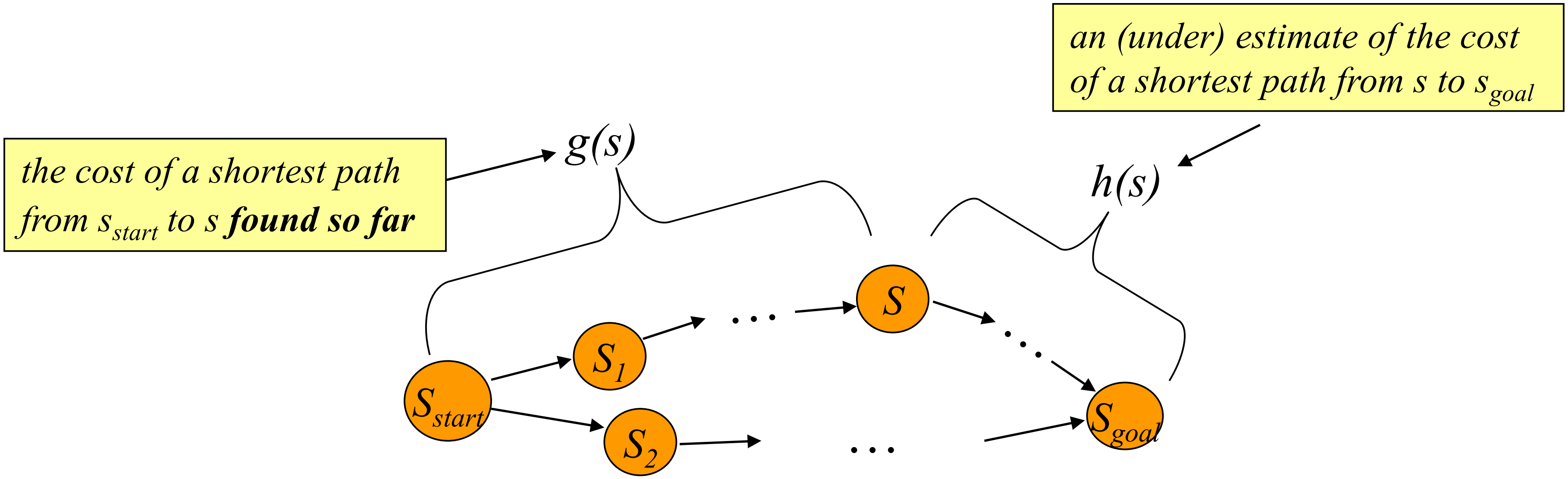
$$s' = \arg\min_{s'' \in pred(s)} (g(s'') + c(s'', s))$$



A* Search [Hart, Nillson, Raphael, '68]

- Computes optimal g-values for relevant states

at any point of time:



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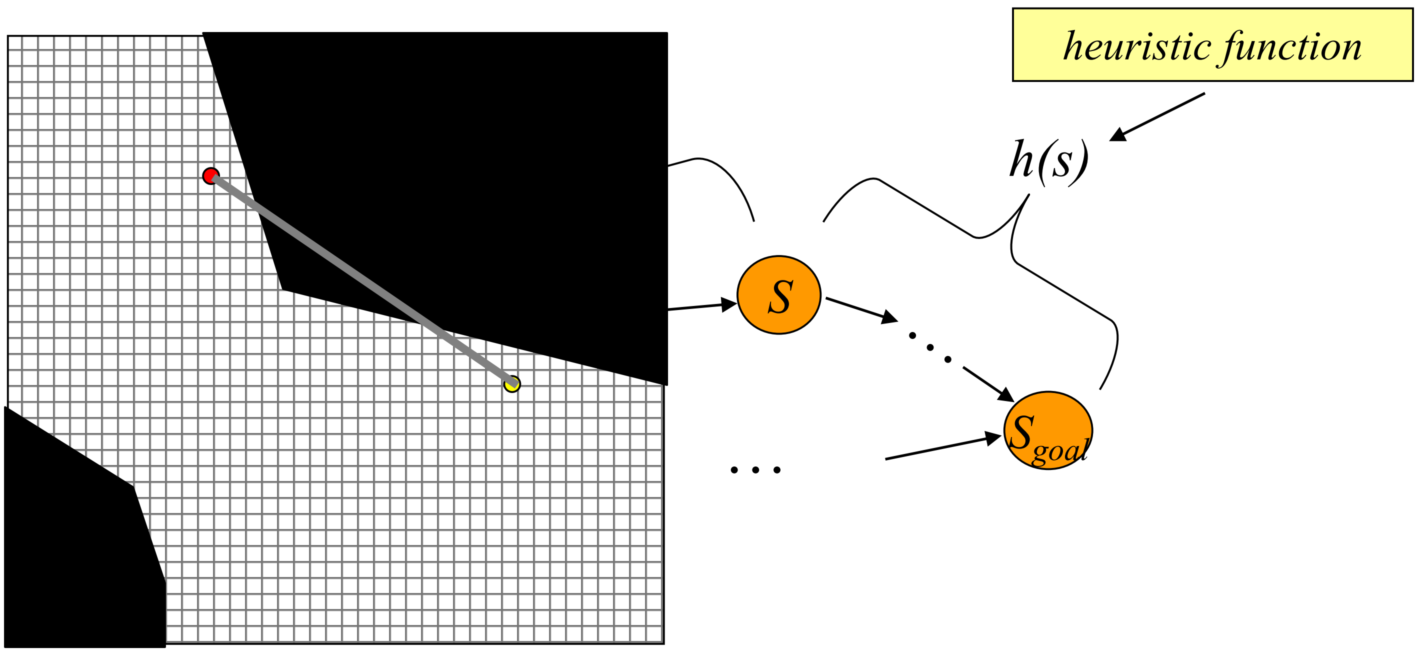


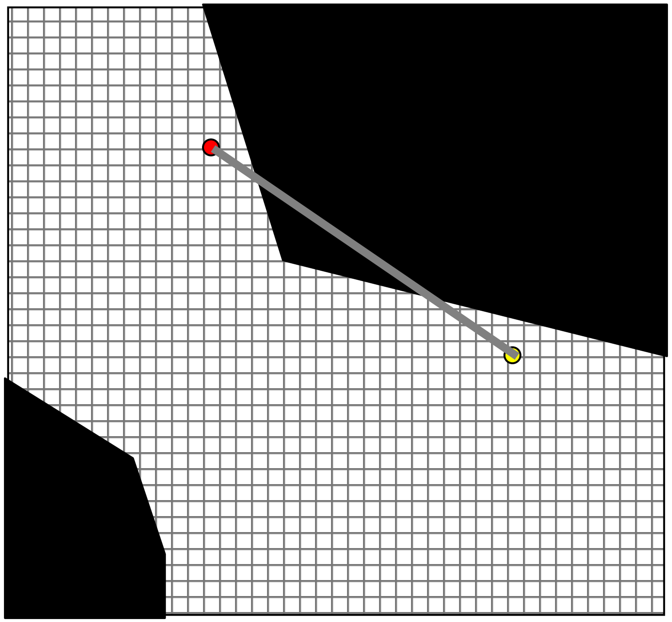
one popular heuristic function – Euclidean distance

A* Search

minimal cost from s to s_{goal}

- Heuristic function must be:
  - admissible: for every state s, $h(s) \leq c^*(s, s_{goal})$
  - consistent (satisfy triangle inequality):

    $h(s_{goal}, s_{goal}) = 0$ *and for every* $s \neq s_{goal}$, $h(s) \leq c(s, succ(s)) + h(succ(s))$
  - admissibility <u>provably</u> follows from consistency and often (<u>not always</u>) consistency follows from admissibility



A* Search

- Computes optimal g-values for relevant states

Main function

$g(s_{start}) = 0$; all other g-values are infinite; $OPEN = \{s_{start}\}$;

ComputePath();

publish solution;

set of candidates for expansion

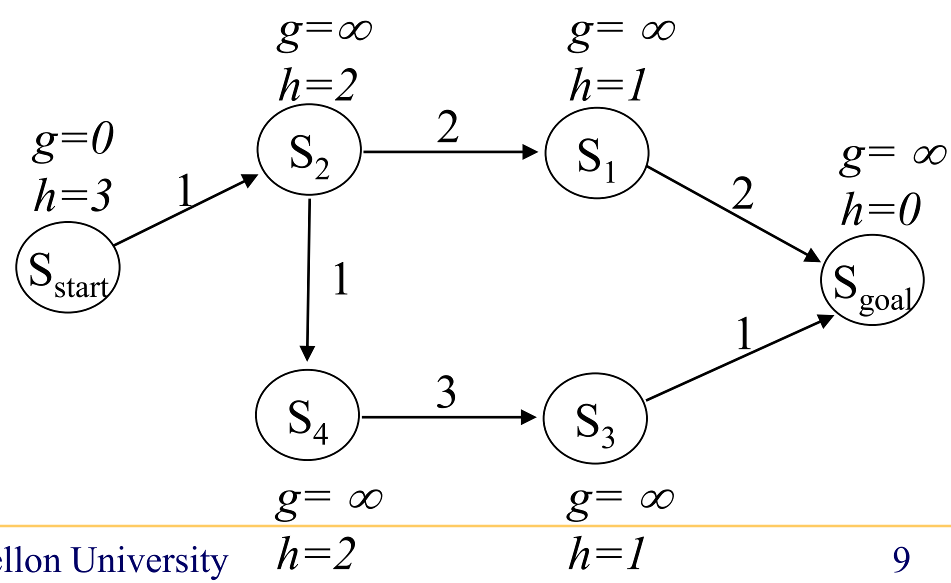
ComputePath function

while(s_{goal} is not expanded and $OPEN \neq 0$)

  remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

  expand s;

for every expanded state
g(s) is optimal
(if heuristics are consistent)



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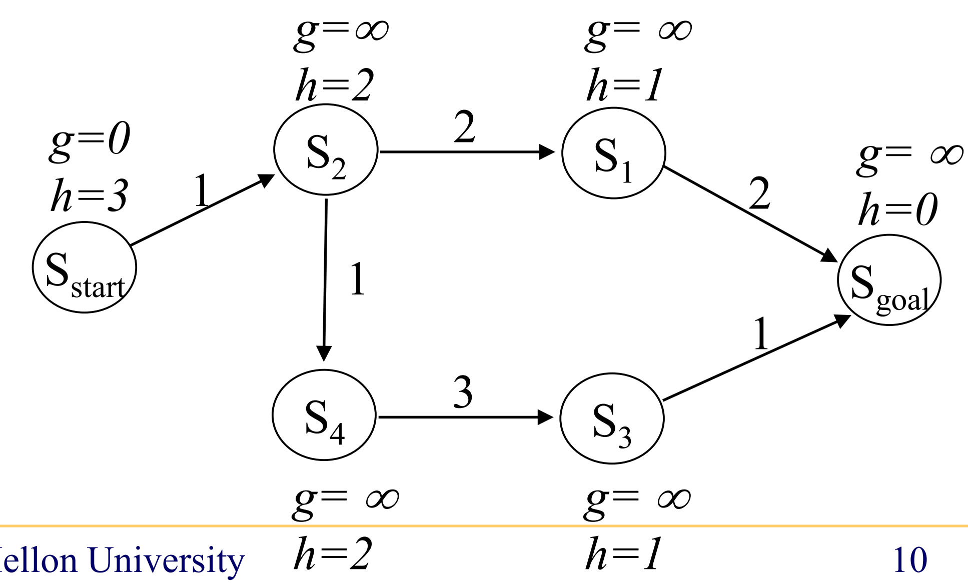

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    if $g(s') > g(s) + c(s,s')$

      $g(s') = g(s) + c(s,s')$;

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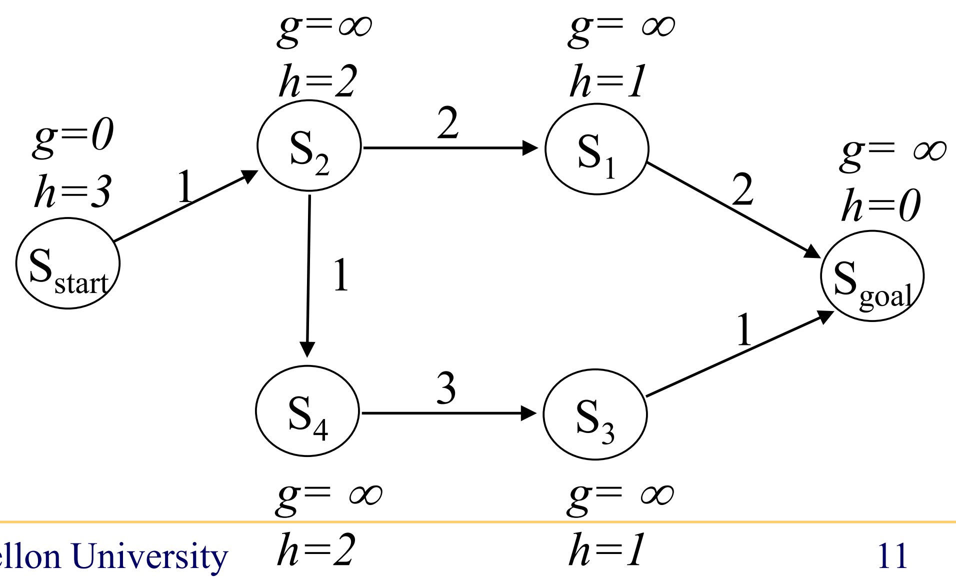
set of states that have already been expanded

tries to decrease $g(s')$ using the found path from s_{start} to s



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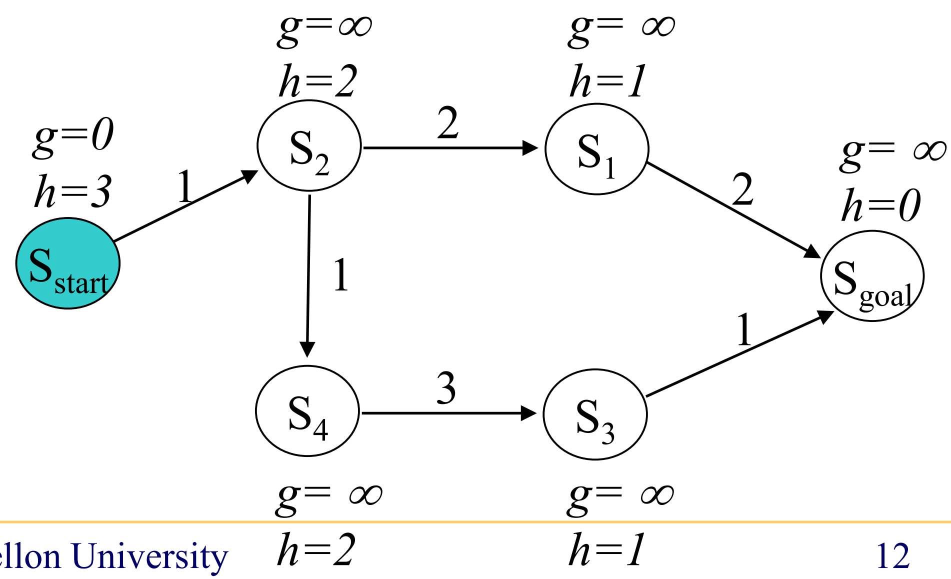
next state to expand: s_{start}



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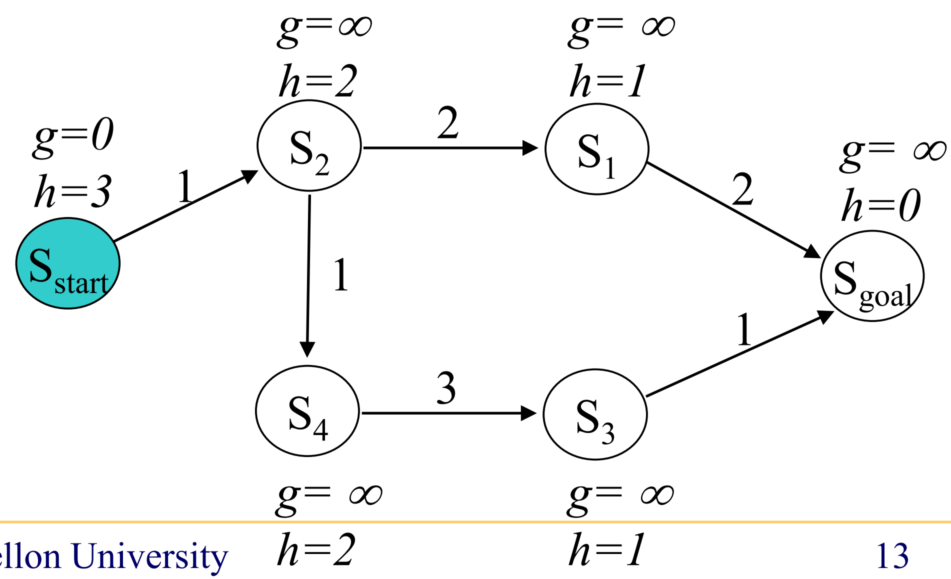
$g(s_2) > g(s_{start}) + c(s_{start}, s_2)$



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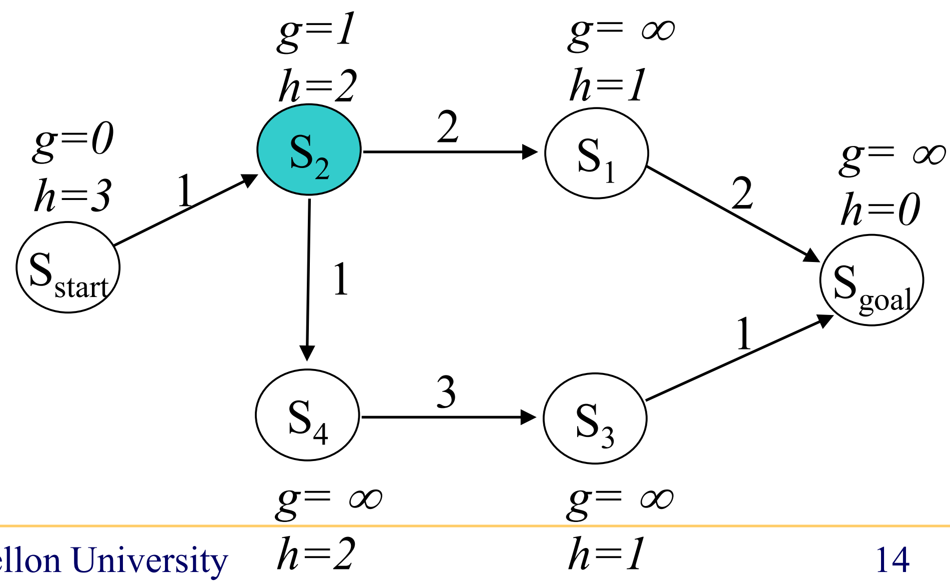

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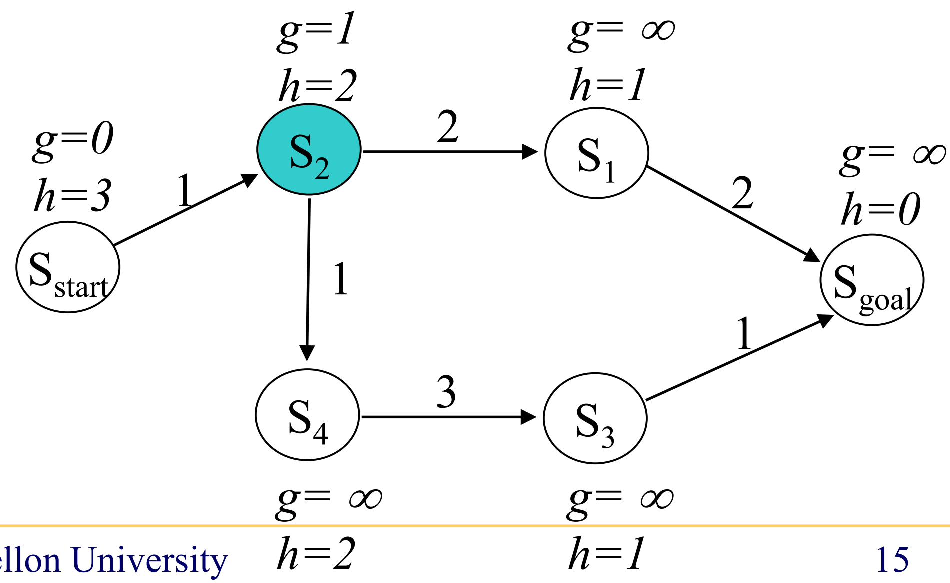
$CLOSED = \{s_{start}\}$

$OPEN = \{s_2\}$

next state to expand: s_2



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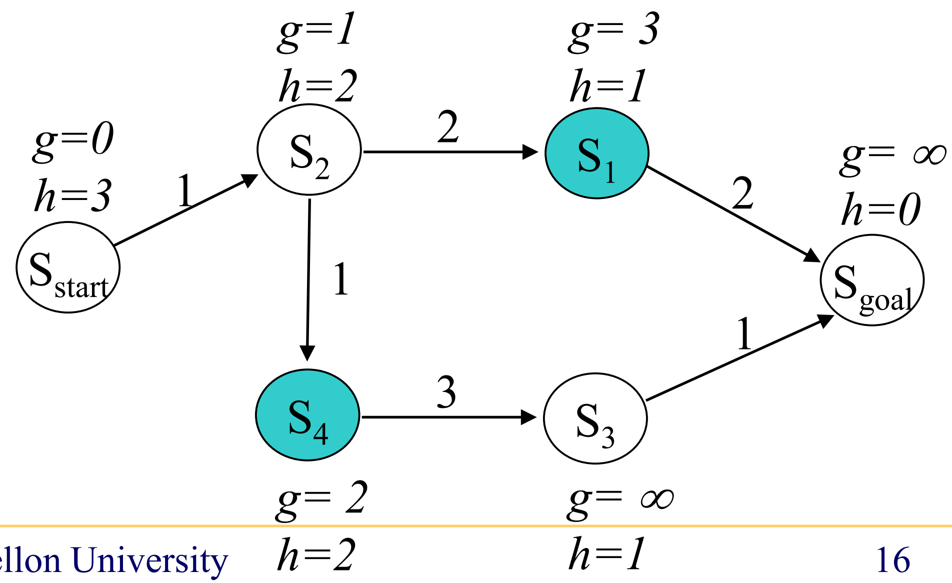
$OPEN = \{s_1, s_4\}$

next state to expand: s_1



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if $g(s') > g(s) + c(s,s')$

$g(s') = g(s) + c(s,s')$;

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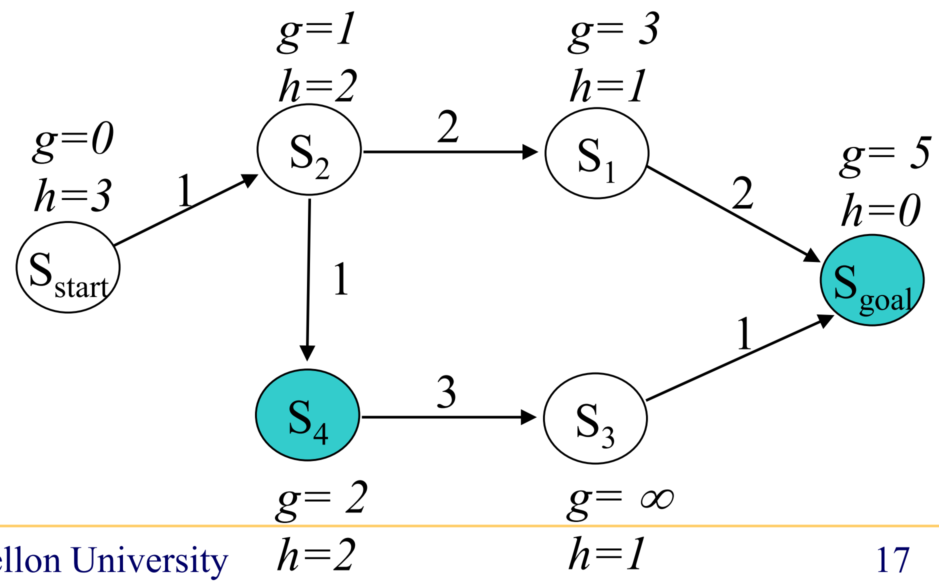
$CLOSED = \{s_{start}, s_2, s_1\}$

$OPEN = \{s_4, s_{goal}\}$

next state to expand: s_4



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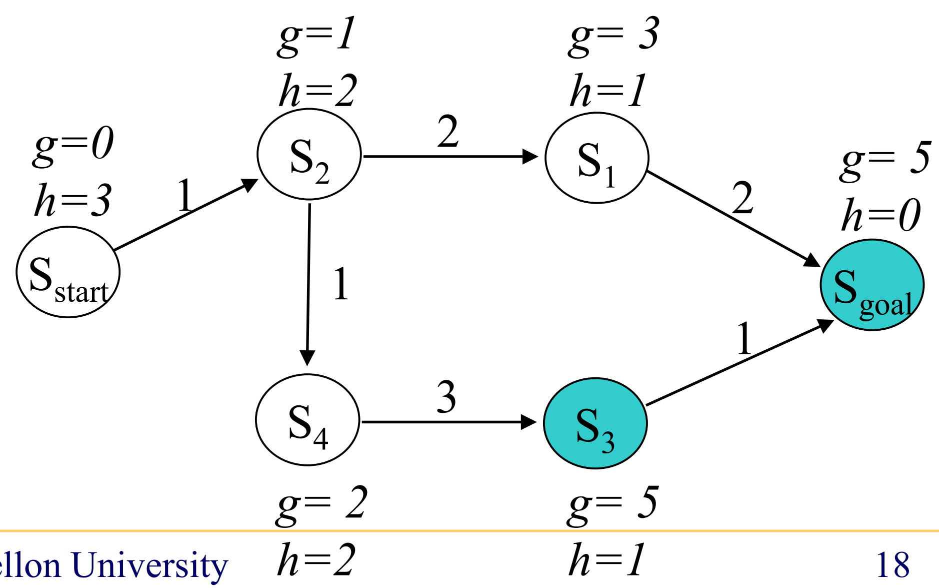
$OPEN = \{s_3, s_{goal}\}$

next state to expand: s_{goal}



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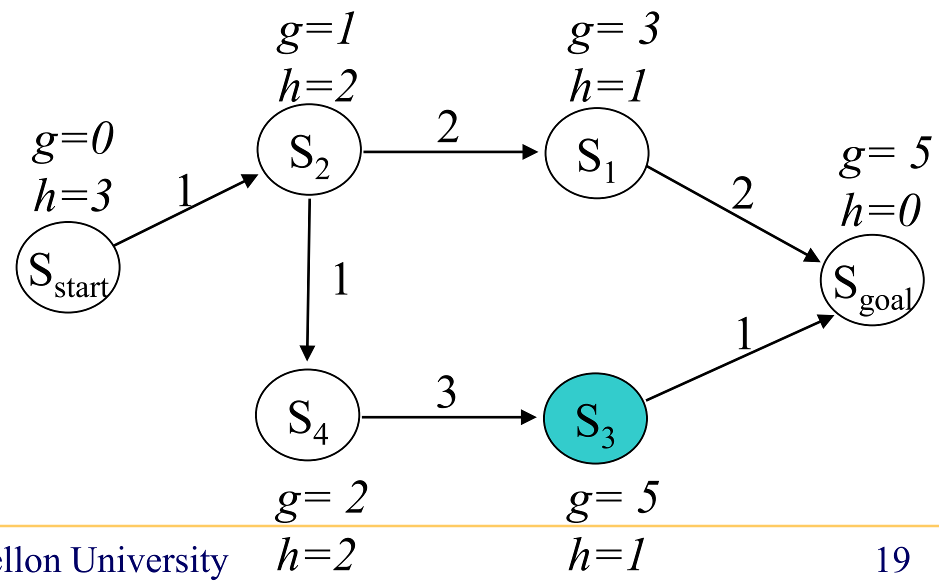
      $g(s') = g(s) + c(s,s')$;
      insert s' into $OPEN$;

$CLOSED = \{s_{start}, s_2, s_1, s_4, s_{goal}\}$
$OPEN = \{s_3\}$
done



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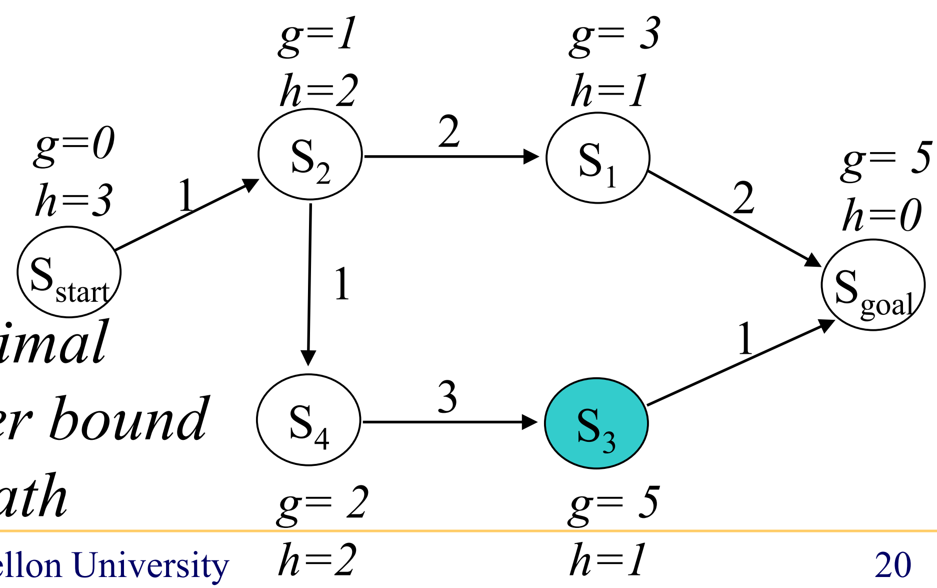

for every expanded state g(s) is optimal

for every other state g(s) is an upper bound

we can now compute a least-cost path

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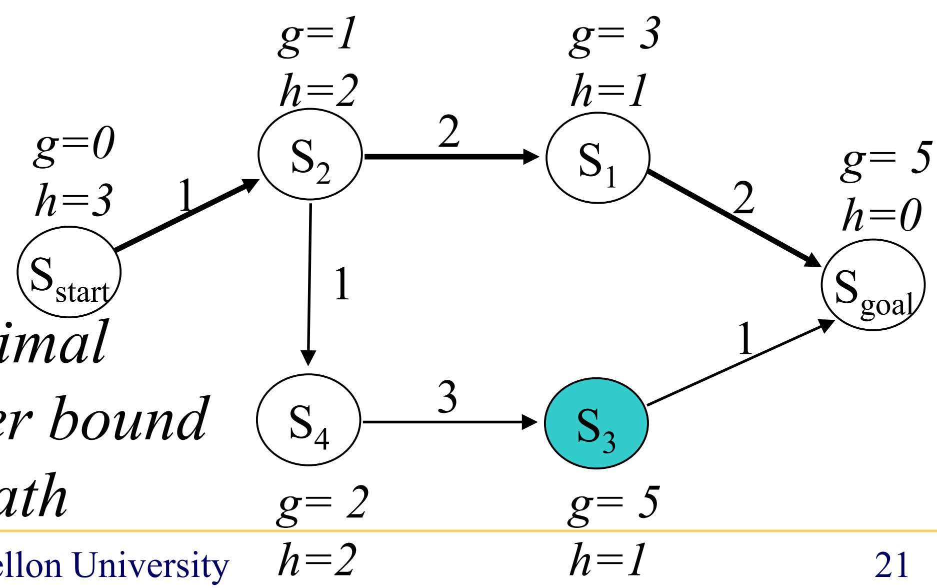

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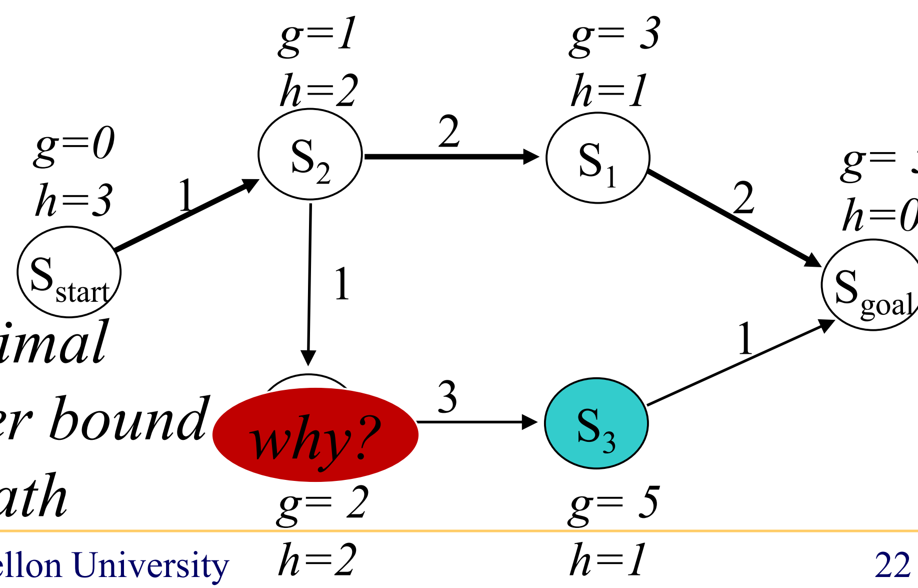

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we can now compute a least-cost path

A* Search

- Is guaranteed to return an optimal path (in fact, for every expanded state) – optimal in terms of the solution

- Performs provably minimal number of state expansions required to guarantee optimality – optimal in terms of the computations

A* Search

- Is guaranteed to return an optimal path (in fact, for every expanded state) – optimal in terms of the solution

Sketch of proof by induction for h = 0:

assume all previously expanded states have optimal g-values

next state to expand is s: f(s) = g(s) – min among states in OPEN

OPEN separates expanded states from never seen states

thus, path to s via a state in OPEN or an unseen state will be worse than g(s) (assuming positive costs)

Effect of the Heuristic Function

- A* Search: expands states in the order of $f = g+h$ values

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Sketch of proof of optimality by induction for consistent h:

1. *assume all previously expanded states have optimal g-values*
2. *next state to expand is s: $f(s) = g(s)+h(s)$ – min among states in OPEN*
3. *assume g(s) is suboptimal*
4. *then there must be at least one state s' on an optimal path from start to s such that it is in OPEN but wasn't expanded*
5. $g(s') + h(s') \geq g(s)+h(s)$
6. *but* $g(s') + c^*(s',s) < g(s) \Rightarrow$

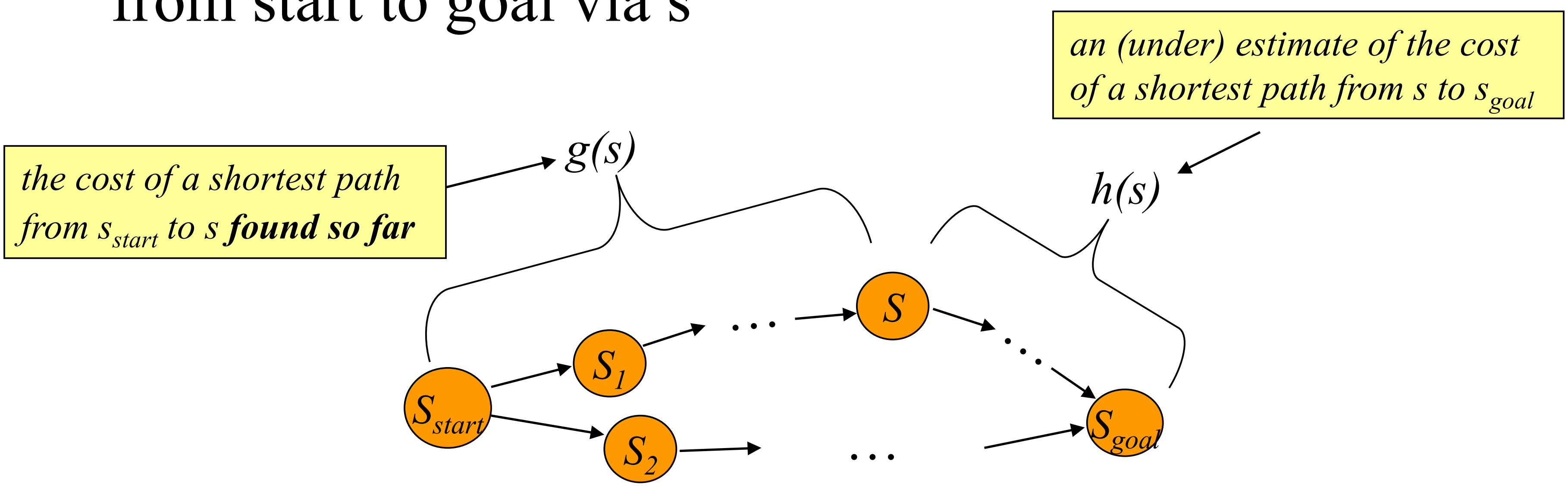
   $$g(s') + c^*(s',s) + h(s) < g(s) + h(s) \Rightarrow$$

   $$g(s') + h(s') < g(s) + h(s)$$
7. *thus it must be the case that g(s) is optimal*

Effect of the Heuristic Function

- A* Search: expands states in the order of $f = g+h$ values
- Dijkstra's: expands states in the order of $f = g$ values (pretty much)

- Intuitively: f(s) – estimate of the cost of a least cost path from start to goal via s



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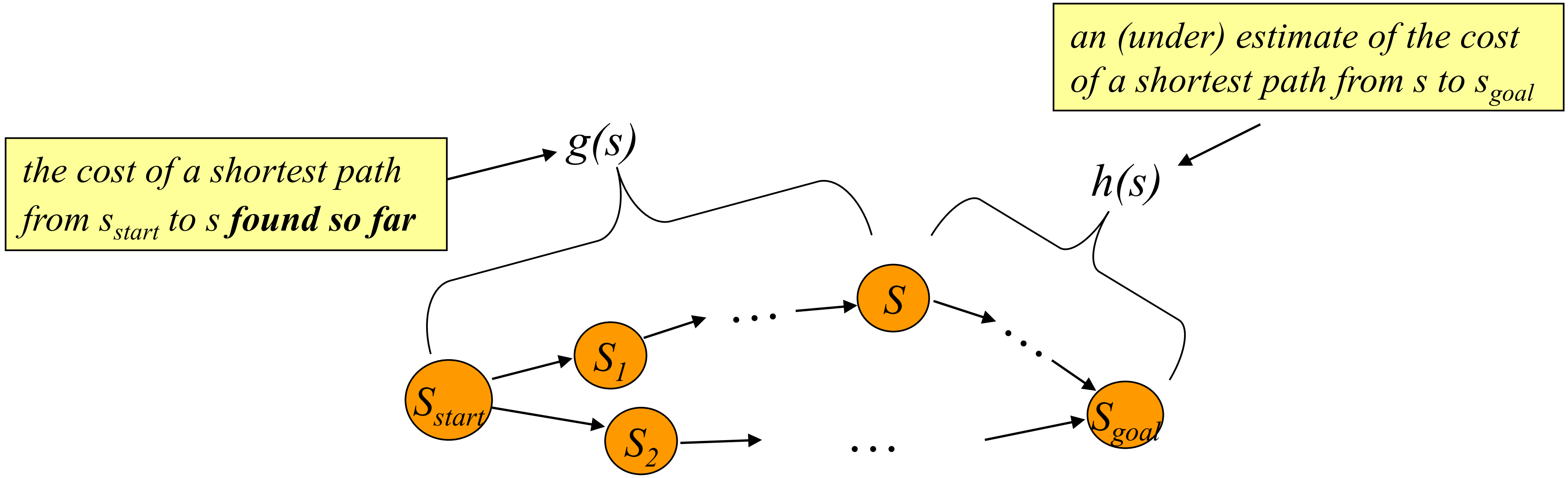
- **Weighted A*:** expands states in the order of $f = g+\varepsilon h$ values, $\varepsilon > 1$ = bias towards states that are closer to goal



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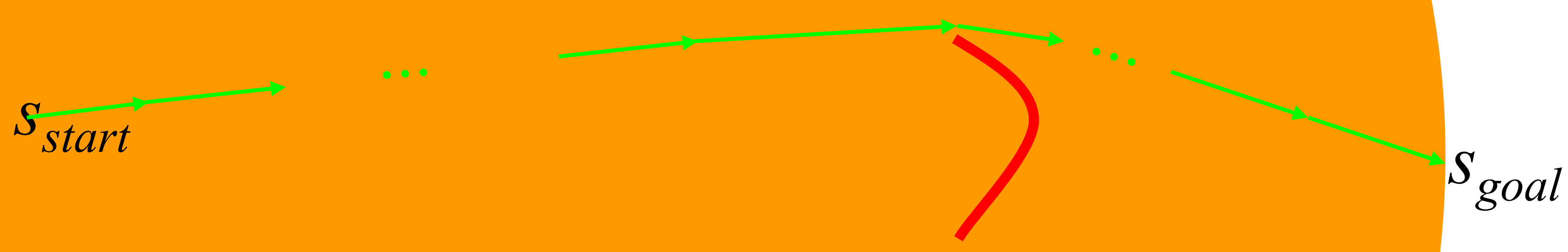
What are the states expanded?



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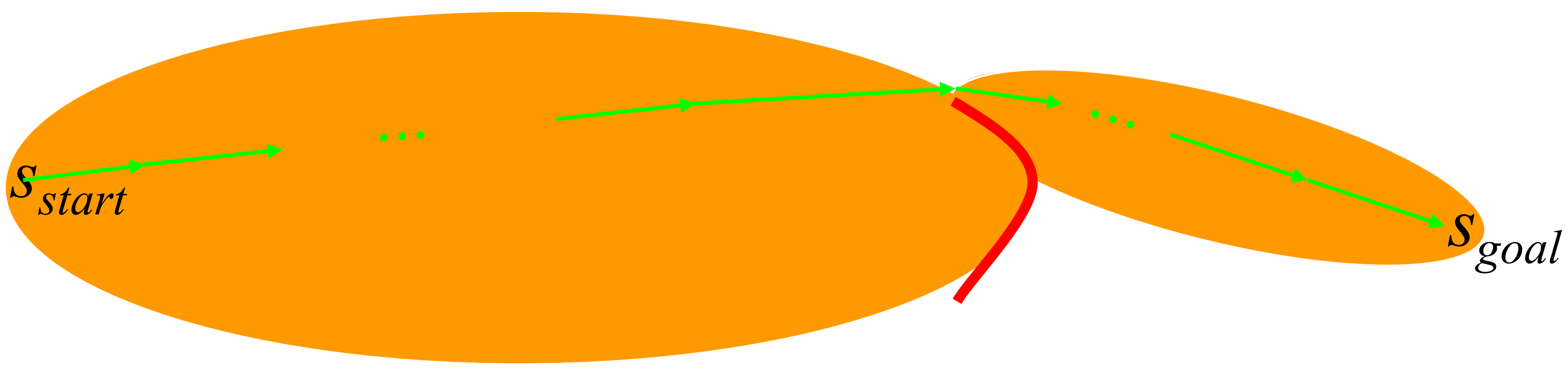

Effect of the Heuristic Function

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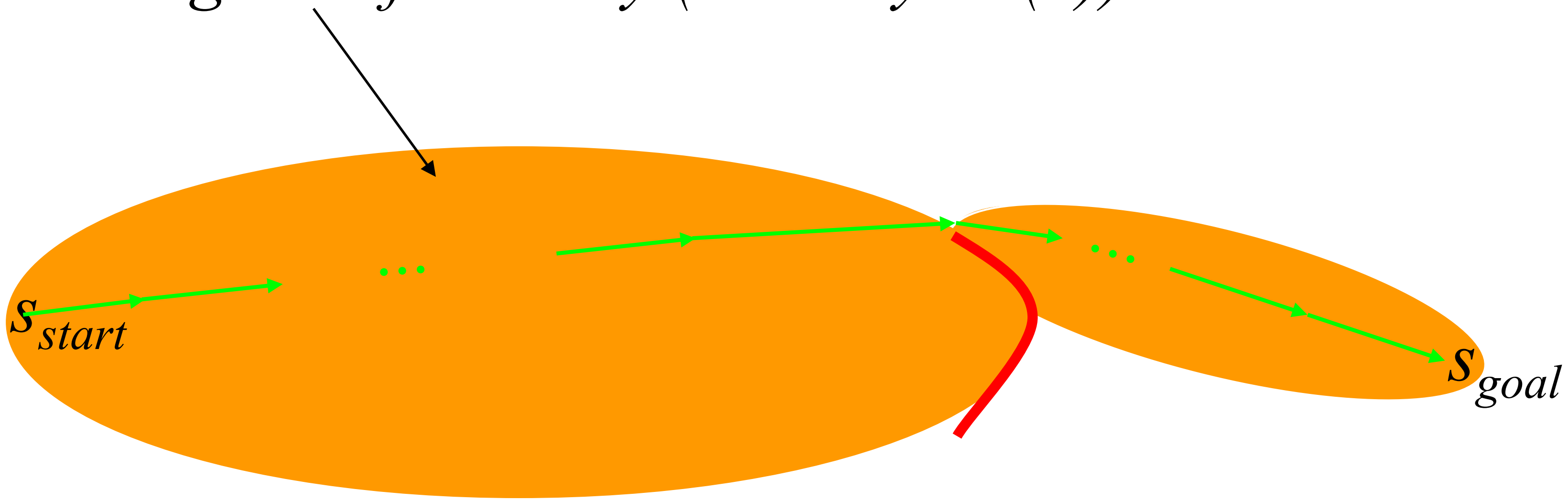
for large problems this results in A quickly running out of memory (memory: O(n))*



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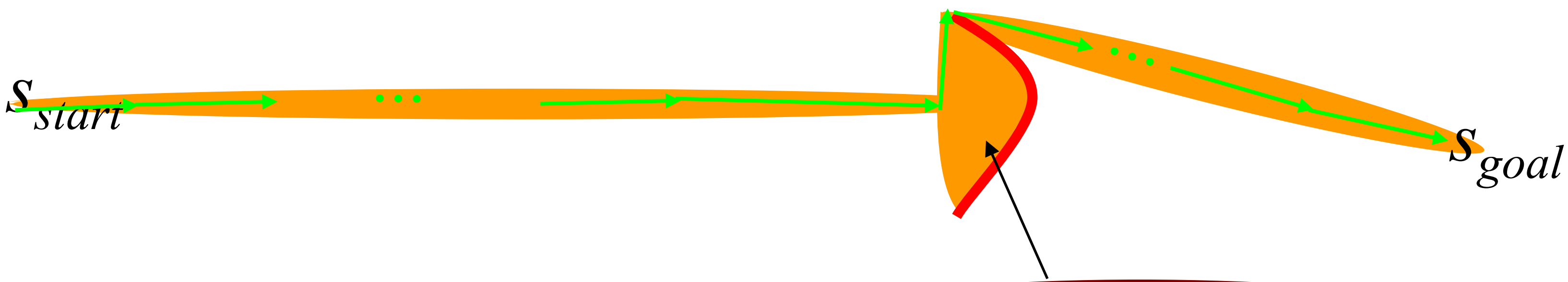

key to finding solution fast:
shallow minima for h(s)-h(s) function*

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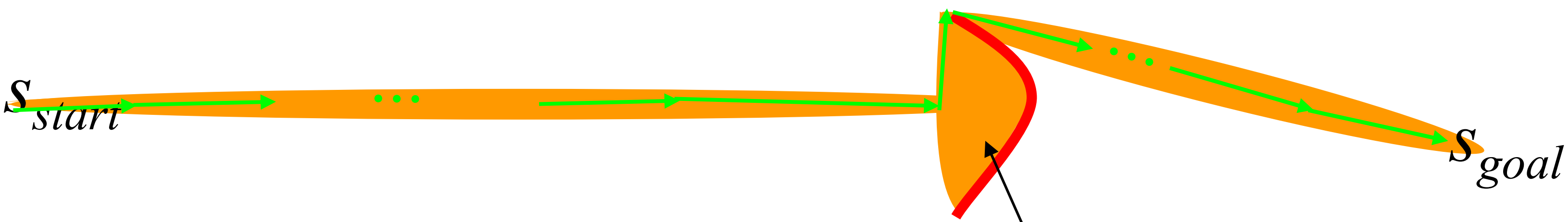
what states are expanded?

No one knows. Topic for research.



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- Weighted A* Search:
  - trades off optimality for speed
  - ε-suboptimal:

    $cost(solution) \leq \varepsilon \cdot cost(optimal\ solution)$
  - in many domains, it has been shown to be orders of magnitude faster than A*
  - research becomes to develop a heuristic function that has shallow local minima

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- Weighted A* Search
  - with re-expansions (no Closed List) [Pohl, '70]
  - without re-expansions (with Closed List) [Likhachev et al., '04]
    - same sub-optimality guarantees <u>but no more than 1 expansion per state</u>

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Is it guaranteed to expand no more states than A?*

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- Searches from goal towards states
- g-values are cost-to-goals

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What needs to be changed?

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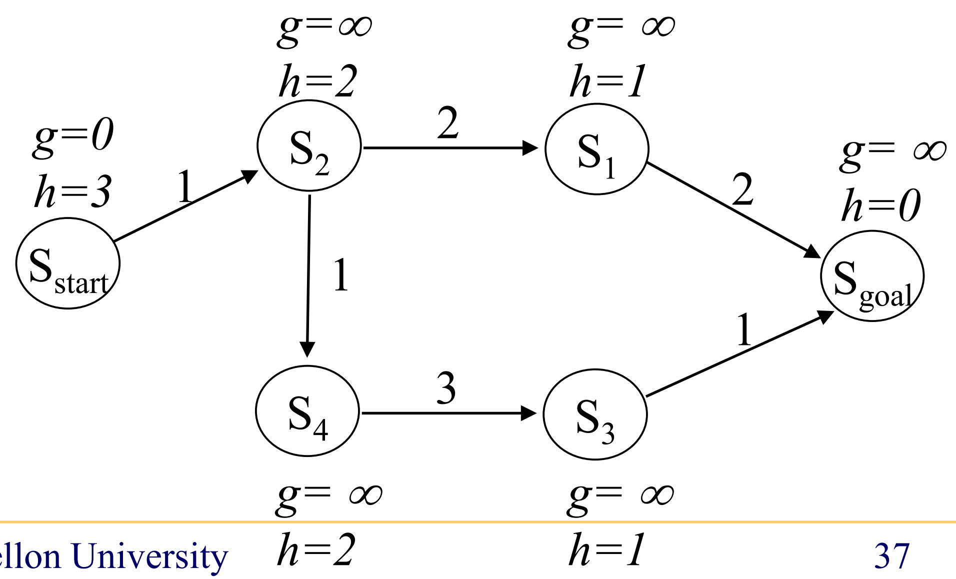

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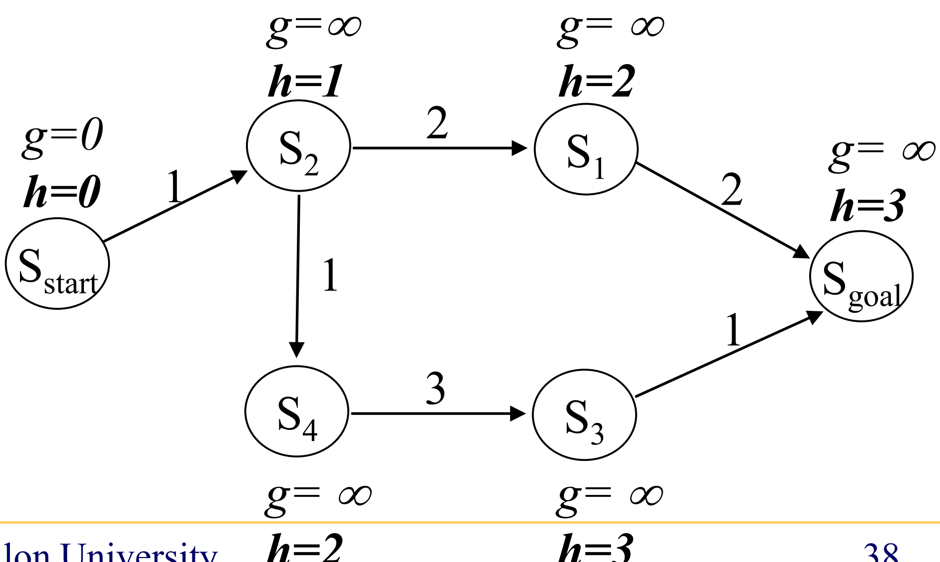

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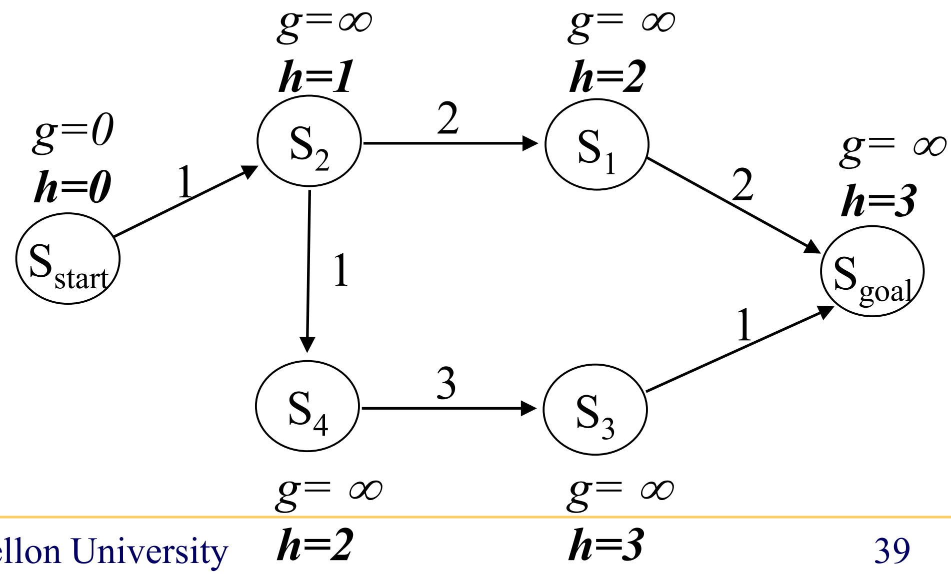

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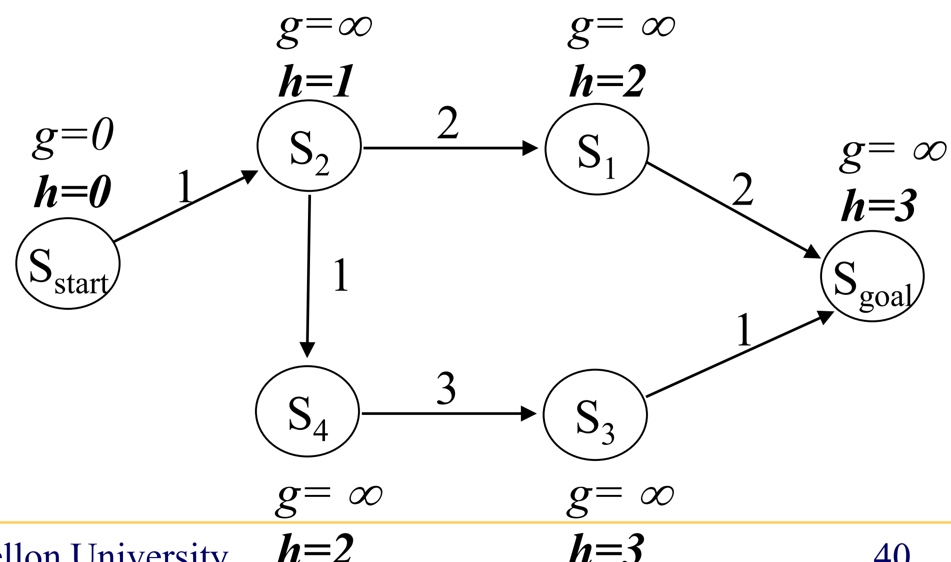
insert s into $CLOSED$;

for every **predecessor** s' of s such that s' not in $CLOSED$

if $g(s') > \boldsymbol{c(s',s) + g(s)}$

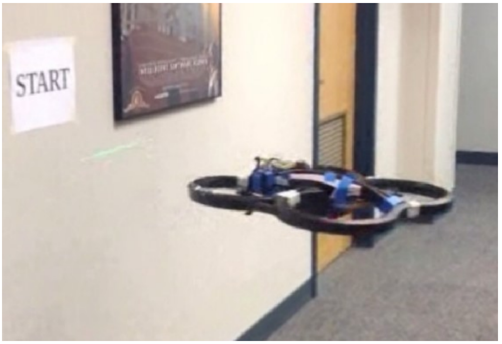
$g(s') = \boldsymbol{c(s',s) + g(s)}$;

insert s' into $OPEN$;



Using A* to Compute a Policy

- Imagine planning for the agent that can easily deviate off the path



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Using A* to Compute a Policy

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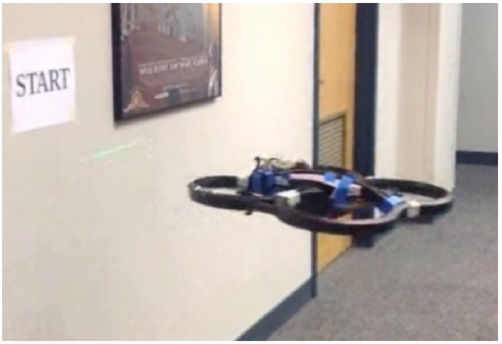


- Can A* compute least-cost paths from **all** the states of interest?
  - Run Backward A* search until all states of interest have been expanded

Using A* to Compute a Policy

- Backward A* search to compute least-cost paths for all states $s \in \Phi$

ComputePath function

while(**at least one state in *Φ* hasn't been expanded** and $OPEN \neq 0$)

remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

insert s into $CLOSED$;

for every predecessor s' of s such that s' not in $CLOSED$

if $g(s') > c(s',s) + g(s)$

$g(s') = c(s',s) + g(s);$

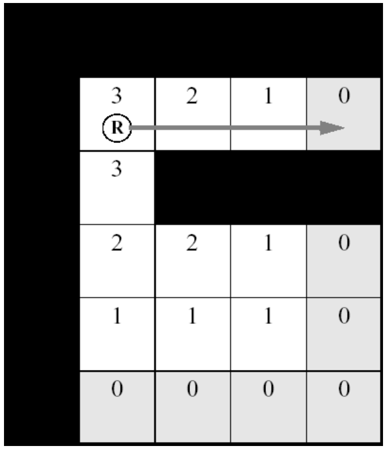
insert s' into $OPEN$;

- Guaranteed to compute least-cost paths for all $s \in \Phi$ that can reach goal

Why?

Support for Multiple Goal Candidates

- How to compute a least-cost path to any one of the possible goals?
  - Example 1: Computing a least-cost path to a parking spot given multiple parking spaces
  - Example 2: Greedy mapping (explore the map by always moving to the closest cell that hasn't been visited yet)



Cells that haven't been visited yet shown in grey

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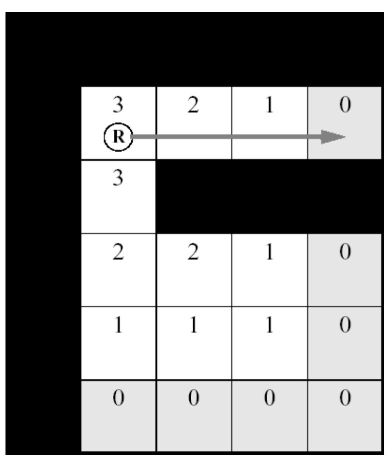

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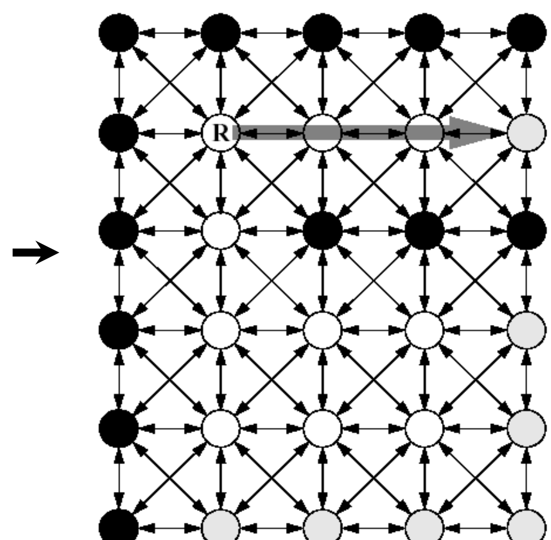

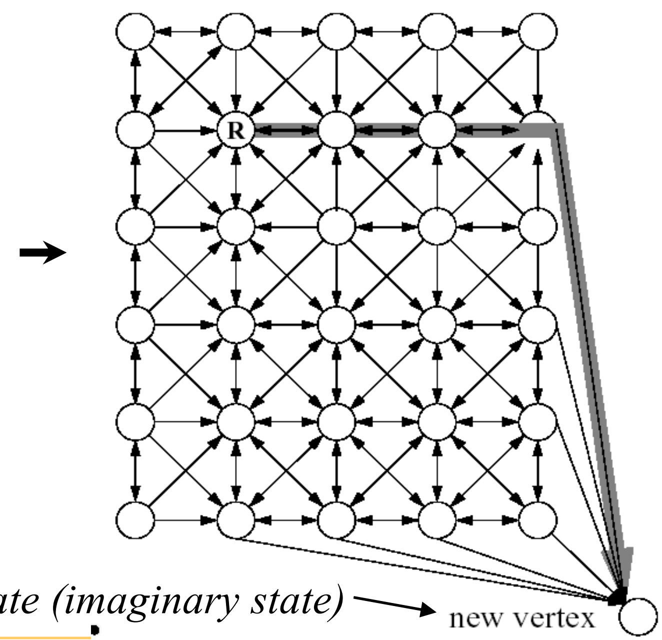

A goal state (imaginary state)

Support for Time-consuming Edge Evaluations

- Lazy weighted A* [Cohen et al., ‘14]
  - use lower bounds on edgecosts in computing g-values
  - when selected for expansion, evaluate the cost of the transition from the predecessor
    - if the same as the lower bound, then expand
    - Otherwise, re-insert back into the queue with the new g-value

Memory Issues

- A* does provably minimum number of expansions (O(n)) for finding a provably optimal solution
- Memory requirements of A* (O(n)) can be improved though
- Memory requirements of weighted A* are often but not always better

Memory Issues

- Alternatives:
  - Depth-First Search (w/o coloring all expanded states):
    - explore each every possible path at a time avoiding looping and keeping in the memory only the best path discovered so far
    - Complete and optimal (assuming finite state-spaces)
    - Memory: $O(bm)$, where b – max. branching factor, m – max. pathlength
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What if goal is few steps away in a huge state-space?

Memory Issues

- Alternatives:
  - IDA* (Iterative Deepening A*) [Korf, '85]
    1. set $f_{max} = 1$ (or some other small value)
    2. execute (previously explained) DFS that does not expand states with $f > f_{max}$
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    - Complete and optimal in any state-space (with positive costs)
    - Memory: $O(bl)$, where b – max. branching factor, l – length of optimal path
    - Complexity: $O(kb^l)$, where k is the number of times DFS is called