Markov Systems with Rewards, Markov Decision Processes

Manuela Veloso

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Where We Are and Outline

- Planning
  - Deterministic state, preconditions, effects
  - Uncertainty
    - Conditional planning, conformant planning, nondeterministic
- Probabilistic modeling of systems with uncertainty and rewards
- Modeling probabilistic systems with control, i.e., action selection
- Reinforcement learning

Axioms of Probability

- Let A be a proposition about the world
- P(A) = probability proposition A is true

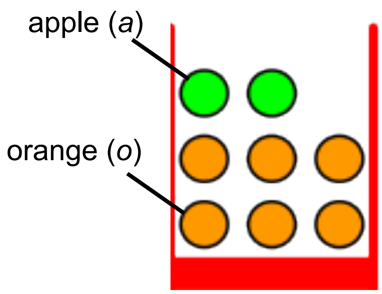
- 0 <= P(A) <= 1
- P(True) = 1
- P(False) = 0
- P(A or B) = P(A) + P(B) – P(A and B)

Random Variables

- Random Variables: variables in probability to capture phenomena
- A random variable has a **domain of values** it can take on.

- Probability distribution function represents **probability of each value**

Example – Pick Fruit from Basket



- Random variable: F
- Domain: a, o
- PDF:
  - p(F = a) = ¼
  - p(F = o) = ¾

Expectation

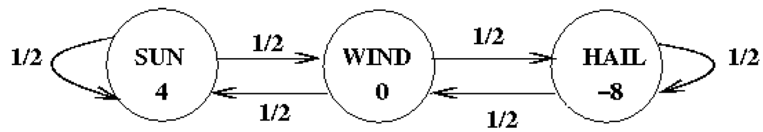
- The **expected value** of a function of a random variable is the weighted average of the probability distribution over outcomes
- Example: expected time of wait for elevator

• Time:	5mn	2mn	0.5mn
• Probability:	0.2	0.7	0.1

⇨

5 x 0.2 + 2 x 0.7+ 0.5 x 0.1 = 2.45mn

Example – Markov System with Reward



• States
• Rewards in states
• Probabilistic transitions between states
• Markov: transitions only depend on current state

Markov Systems with Rewards

- Finite set of n states, s_i
- Probabilistic state matrix, P, p_{ij}
- "Goal achievement" - Reward for each state, r_i
- Discount factor - γ
- Process/observation:
  - – Assume start state s_i
  - – Receive immediate reward r_i
  - – Move, or observe a move, randomly to a new state according to the probability transition matrix
  - – Future rewards (of next state) are discounted by γ

Solving a Markov System with Rewards

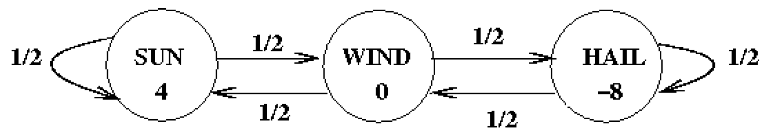
- $V^*(s_i)$ - expected discounted sum of future rewards starting in state s_i

- $V^*(s_i) = r_i + \gamma[p_{i1}V^*(s_1) + p_{i2}V^*(s_2) + \ldots\ p_{in}V^*(s_n)]$

Value Iteration to Solve a Markov System with Rewards

- $V^1(s_i)$ - expected discounted sum of future rewards starting in state s_i *for one step*.
- $V^2(s_i)$ - expected discounted sum of future rewards starting in state s_i *for two steps*.
- ...
- $V^k(s_i)$ - expected discounted sum of future rewards starting in state s_i *for k steps*.
- As $k \rightarrow \infty\ V^k(s_i) \rightarrow V^*(s_i)$
- Stop when difference of $k + 1$ and k values is smaller than some $\in$.

3-State Example



3-State Example: Values $\gamma = 0.5$

Iteration	SUN	WIND	HAIL
0	0	0	0
1	4	0	-8
2	5.0	-1.0	-10.0
3	5.0	-1.25	-10.75
4	4.9375	-1.4375	-11.0
5	4.875	-1.515625	-11.109375
6	4.8398437	-1.5585937	-11.15625
7	4.8203125	-1.5791016	-11.178711
8	4.8103027	-1.5895996	-11.189453
9	4.805176	-1.5947876	-11.194763
10	4.802597	-1.5973969	-11.197388
11	4.8013	-1.5986977	-11.198696
12	4.8006506	-1.599349	-11.199348
13	4.8003254	-1.5996745	-11.199675
14	4.800163	-1.5998373	-11.199837
15	4.8000813	-1.5999185	-11.199919

3-State Example: Values γ = 0.9

Iteration	SUN	WIND	HAIL
0	0	0	0
1	4	0	-8
2	5.8	-1.8	-11.6
3	5.8	-2.6100001	-14.030001
4	5.4355	-3.7035	-15.488001
5	4.7794	-4.5236254	-16.636175
6	4.1150985	-5.335549	-17.521912
7	3.4507973	-6.0330653	-18.285858
8	2.8379793	-6.6757774	-18.943516
9	2.272991	-7.247492	-19.528683
...	...	...	...
50	-2.8152928	-12.345073	-24.633476
51	-2.8221645	-12.351946	-24.640347
52	-2.8283496	-12.3581295	-24.646532
...	...	...	...
86	-2.882461	-12.412242	-24.700644
87	-2.882616	-12.412397	-24.700798
88	-2.8827558	-12.412536	-24.70094

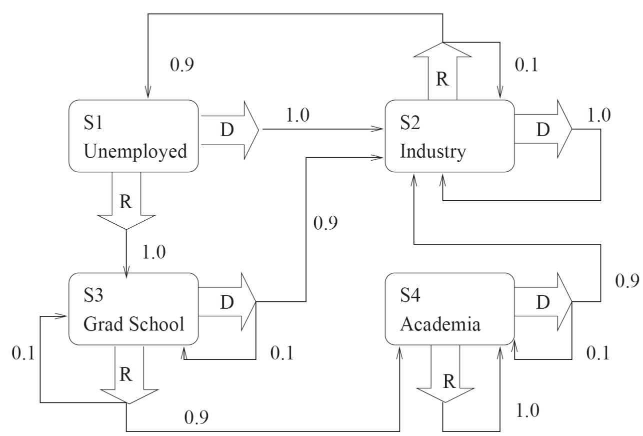
3-State Example: Values γ = 0.2

Iteration	SUN	WIND	HAIL
0	0	0	0
1	4	0	-8
2	4.4	-0.4	-8.8
3	4.4	-0.44000003	-8.92
4	4.396	-0.452	-8.936
5	4.3944	-0.454	-8.9388
6	4.39404	-0.45443997	-8.93928
7	4.39396	-0.45452395	-8.939372
8	4.393944	-0.4545412	-8.939389
9	4.3939404	-0.45454454	-8.939393
10	4.3939395	-0.45454526	-8.939394
11	4.3939395	-0.45454547	-8.939394
12	4.3939395	-0.45454547	-8.939394

Markov Decision Processes

- Finite set of states, $s_1, \ldots, s_n$
- Finite set of actions, $a_1, \ldots, a_m$
- Probabilistic state,action transitions:
  $p_{ij}^k = \text{prob}(\text{next} = s_j \mid \text{current} = s_i \text{ and take action } a_k)$
- Markov assumption: State transition function only dependent on current state, not on the "history" of how the state was reached.
- Reward for each state, $r_1, \ldots, r_n$
- Process:
  - Start in state s_i
  - Receive immediate reward r_i
  - Choose action $a_k \in A$
  - Change to state s_j with probability p_{ij}^k.
  - Discount future rewards

Nondeterministic Example



Reward and discount factor to be decided.
Note the need to have a finite set of states and actions.
Note the need to have all transition probabilties.

Solving an MDP

- Find an action to apply to each state.
- A policy is a mapping from states to actions.
- Optimal policy - for every state, there is no other action that gets a higher sum of discounted future rewards.
- For every MDP there exists an optimal policy.
- Solving an MDP is finding an optimal policy.
- A specific policy converts an MDP into a plain Markov system with rewards.

Value Iteration

- $V^*(s_i)$ - expected discounted future rewards, if we start from state s_i, and we follow the optimal policy.
- Compute V^* with value iteration:
  - $V^k(s_i)$ = maximum possible future sum of rewards starting from state s_i for k steps.
- Bellman's Equation:

$$V^{n+1}(s_i) = \max_k \{r_i + \gamma \sum_{j=1}^{N} p_{ij}^k V^n(s_j)\}$$

- Dynamic programming

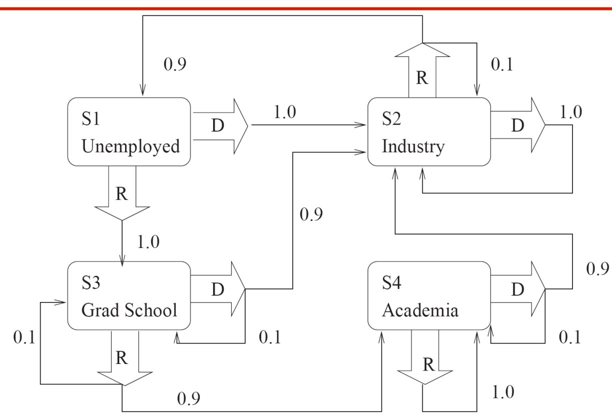
Policy Iteration

- Start with some policy $\pi_0(s_i)$.
- Such policy transforms the MDP into a plain Markov system with rewards.
- Compute the values of the states according to the current policy.
- Update policy:

$$\pi_{k+1}(s_i) = \arg\max_a \{r_i + \gamma \sum_j p_{ij}^a V^{\pi_k}(s_j)\}$$

- Keep computing
- Stop when $\pi_{k+1} = \pi_k$.

Nondeterministic Example



REWARD

S1	S2	S3	S4
0	100	0	10

Nondeterministic Example

$\pi^*(s) = D$, for any s= S1, S2, S3, and S4, $\gamma = 0.9$.

```
V*(S2) = r(S2,D) + 0.9 (1.0 V*(S2))
V*(S2) = 100 + 0.9 V*(S2)
V*(S2) = 1000.

V*(S1) = r(S1,D) + 0.9 (1.0 V*(S2))
V*(S1) = 0 + 0.9 x 1000
V*(S1) = 900.

V*(S3) = r(S3,D) + 0.9 (0.9 V*(S2) + 0.1 V*(S3))
V*(S3) = 0 + 0.9 (0.9 x 1000 + 0.1 V*(S3))
V*(S3) = 81000/91.

V*(S4) = r(S4,D) + 0.9 (0.9 V*(S2) + 0.1 V*(S4))
V*(S4) = 40 + 0.9 (0.9 x 1000 + 0.1 V*(S4))
V*(S4) = 85000/91.
```

Markov Models

- Plan is a ***Policy***
  - ***Stationary***: Best action is fixed
  - ***Non-stationary***: Best action depends on time
- States can be *discrete*, ***continuous***, or ***hybrid***

	Passive	**Controlled**
Fully Observable	Markov Systems with Rewards	MDP
Hidden State	HMM	POMDP
Time Dependent	Semi-Markov	SMDP

Tradeoffs

- **MDPs**
  - + Tractable to solve
  - + Relatively easy to specify
  - – Assumes perfect knowledge of state
- **POMDPs**
  - + Treats all sources of uncertainty uniformly
  - + Allows for taking actions that gain information
  - – Difficult to specify all the conditional probabilities
  - – *Hugely* intractable to solve optimally
- **SMDPs**
  - + General distributions for action durations
  - – Few good solution algorithms

Summary

- Markov Models with Reward
- Value iteration
- Markov Decision Process
- Value Iteration
- Policy Iteration
- Reinforcement Learning