Local Search for Fast Matrix Multiplication

Marijn Heule, Manuel Kauers, and Martina Seidl

THE UNIVERSITY OF TEXAS AT AUSTIN

JOHANNES KEPLER UNIVERSITÄT LINZ

Starting at Carnegie Mellon University in August

SAT 2019 Conference, Lisbon July 9, 2019

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

$$c_{1,1} = a_{1,1} \cdot b_{1,1} + a_{1,2} \cdot b_{2,1}$$
$$c_{1,2} = a_{1,1} \cdot b_{1,2} + a_{1,2} \cdot b_{2,2}$$
$$c_{2,1} = a_{2,1} \cdot b_{1,1} + a_{2,2} \cdot b_{2,1}$$
$$c_{2,2} = a_{2,1} \cdot b_{1,2} + a_{2,2} \cdot b_{2,2}$$

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

$$c_{1,1} = M_1 + M_4 - M_5 + M_7$$
$$c_{1,2} = M_3 + M_5$$
$$c_{2,1} = M_2 + M_4$$
$$c_{2,2} = M_1 - M_2 + M_3 + M_6$$

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

... where

$$\begin{aligned}
M_1 &= (a_{1,1} + a_{2,2}) \cdot (b_{1,1} + b_{2,2}) \\
M_2 &= (a_{2,1} + a_{2,2}) \cdot b_{1,1} \\
M_3 &= a_{1,1} \cdot (b_{1,2} - b_{2,2}) \\
M_4 &= a_{2,2} \cdot (b_{2,1} - b_{1,1}) \\
M_5 &= (a_{1,1} + a_{1,2}) \cdot b_{2,2} \\
M_6 &= (a_{2,1} - a_{1,1}) \cdot (b_{1,1} + b_{1,2}) \\
M_7 &= (a_{1,2} - a_{2,2}) \cdot (b_{2,1} + b_{2,2})
\end{aligned}$$

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

- This scheme needs 7 multiplications instead of 8.

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

- This scheme needs 7 multiplications instead of 8.
- Recursive application allows to multiply $n \times n$ matrices with $\mathcal{O}(n^{\log_2 7})$ operations in the ground ring.

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

- This scheme needs 7 multiplications instead of 8.
- Recursive application allows to multiply $n \times n$ matrices with $\mathcal{O}(n^{\log_2 7})$ operations in the ground ring.
- Let ω be the smallest number so that $n \times n$ matrices can be multiplied using $\mathcal{O}(n^{\omega})$ operations in the ground domain.

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

- This scheme needs 7 multiplications instead of 8.
- Recursive application allows to multiply $n \times n$ matrices with $\mathcal{O}(n^{\log_2 7})$ operations in the ground ring.
- Let ω be the smallest number so that $n \times n$ matrices can be multiplied using $\mathcal{O}(n^{\omega})$ operations in the ground domain.
- Then $2 \leq \omega < 3$. What is the exact value?

Efficient Matrix Multiplication: Theory

- Strassen 1969: $\omega \leq \log_2 7 \leq 2.807$

Efficient Matrix Multiplication: Theory

▸ Strassen 1969:	$\omega \leq \log_2 7 \leq 2.807$
▸ Pan 1978:	$\omega \leq 2.796$
▸ Bini et al. 1979:	$\omega \leq 2.7799$
▸ Schönhage 1981:	$\omega \leq 2.522$
▸ Romani 1982:	$\omega \leq 2.517$
▸ Coppersmith/Winograd 1981:	$\omega \leq 2.496$
▸ Strassen 1986:	$\omega \leq 2.479$
▸ Coppersmith/Winograd 1990:	$\omega \leq 2.376$

Efficient Matrix Multiplication: Theory

- Strassen 1969: $\omega \leq \log_2 7 \leq 2.807$
- Pan 1978: $\omega \leq 2.796$
- Bini et al. 1979: $\omega \leq 2.7799$
- Schönhage 1981: $\omega \leq 2.522$
- Romani 1982: $\omega \leq 2.517$
- Coppersmith/Winograd 1981: $\omega \leq 2.496$
- Strassen 1986: $\omega \leq 2.479$
- Coppersmith/Winograd 1990: $\omega \leq 2.376$
- Stothers 2010: $\omega \leq 2.374$
- Williams 2011: $\omega \leq 2.3728642$
- Le Gall 2014: $\omega \leq 2.3728639$

Efficient Matrix Multiplication: Practice

- Only Strassen's algorithm beats the classical algorithm for reasonable problem sizes.

Efficient Matrix Multiplication: Practice

- Only Strassen's algorithm beats the classical algorithm for reasonable problem sizes.
- Want: a matrix multiplication algorithm that beats Strassen's algorithm for matrices of moderate size.

Efficient Matrix Multiplication: Practice

- Only Strassen's algorithm beats the classical algorithm for reasonable problem sizes.
- Want: a matrix multiplication algorithm that beats Strassen's algorithm for matrices of moderate size.
- Idea: instead of dividing the matrices into 2×2-block matrices, divide them into 3×3-block matrices.

Efficient Matrix Multiplication: Practice

- Only Strassen's algorithm beats the classical algorithm for reasonable problem sizes.
- Want: a matrix multiplication algorithm that beats Strassen's algorithm for matrices of moderate size.
- Idea: instead of dividing the matrices into 2×2-block matrices, divide them into 3×3-block matrices.
- Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

Efficient Matrix Multiplication: Practice

- Only Strassen's algorithm beats the classical algorithm for reasonable problem sizes.
- Want: a matrix multiplication algorithm that beats Strassen's algorithm for matrices of moderate size.
- Idea: instead of dividing the matrices into 2×2-block matrices, divide them into 3×3-block matrices.
- Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?
- Answer: Nobody knows.

The 3*x*3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- naive algorithm: 27

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- naive algorithm: 27
- padd with zeros, use Strassen twice, cleanup: 25

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- naive algorithm: 27
- padd with zeros, use Strassen twice, cleanup: 25
- best known upper bound: 23 (Laderman 1976)

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- naive algorithm: 27
- padd with zeros, use Strassen twice, cleanup: 25
- best known upper bound: 23 (Laderman 1976)
- best known lower bound: 19 (Bläser 2003)

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- naive algorithm: 27
- padd with zeros, use Strassen twice, cleanup: 25
- best known upper bound: 23 (Laderman 1976)
- best known lower bound: 19 (Bläser 2003)
- maximal number of multiplications allowed if we want to beat Strassen: 21 (because $\log_3 21 < \log_2 7 < \log_3 22$).

Laderman's scheme from 1976

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

$$\begin{aligned}
c_{1,1} &= -M_6 + M_{14} + M_{19} \\
c_{2,1} &= M_2 + M_3 + M_4 + M_6 + M_{14} + M_{16} + M_{17} \\
c_{3,1} &= M_6 + M_7 - M_8 + M_{11} + M_{12} + M_{13} - M_{14} \\
c_{1,2} &= M_1 - M_4 + M_5 - M_6 - M_{12} + M_{14} + M_{15} \\
c_{2,2} &= M_2 + M_4 - M_5 + M_6 + M_{20} \\
c_{3,2} &= M_{12} + M_{13} - M_{14} - M_{15} + M_{22} \\
c_{1,3} &= -M_6 - M_7 + M_9 + M_{10} + M_{14} + M_{16} + M_{18} \\
c_{2,3} &= M_{14} + M_{16} + M_{17} + M_{18} + M_{21} \\
c_{3,3} &= M_6 + M_7 - M_8 - M_9 + M_{23}
\end{aligned}$$

Laderman's scheme from 1976

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

where ...

$$M_1 = (-a_{1,1} + a_{1,2} + a_{1,3} - a_{2,1} + a_{2,2} + a_{3,2} + a_{3,3}) \cdot b_{2,2}$$
$$M_2 = (a_{1,1} + a_{2,1}) \cdot (b_{1,2} + b_{2,2})$$
$$M_3 = a_{2,2} \cdot (b_{1,1} - b_{1,2} + b_{2,1} - b_{2,2} - b_{2,3} + b_{3,1} - b_{3,3})$$
$$M_4 = (-a_{1,1} - a_{2,1} + a_{2,2}) \cdot (-b_{1,1} + b_{1,2} + b_{2,2})$$
$$M_5 = (-a_{2,1} + a_{2,2}) \cdot (-b_{1,1} + b_{1,2})$$
$$M_6 = -a_{1,1} \cdot b_{1,1}$$
$$M_7 = (a_{1,1} + a_{3,1} + a_{3,2}) \cdot (b_{1,1} - b_{1,3} + b_{2,3})$$
$$M_8 = (a_{1,1} + a_{3,1}) \cdot (-b_{1,3} + b_{2,3})$$
$$M_9 = (a_{3,1} + a_{3,2}) \cdot (b_{1,1} - b_{1,3})$$

Laderman's scheme from 1976

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

where ...

$$M_{10} = (a_{1,1} + a_{1,2} - a_{1,3} - a_{2,2} + a_{2,3} + a_{3,1} + a_{3,2}) \cdot b_{2,3}$$
$$M_{11} = (a_{3,2}) \cdot (-b_{1,1} + b_{1,3} + b_{2,1} - b_{2,2} - b_{2,3} - b_{3,1} + b_{3,2})$$
$$M_{12} = (a_{1,3} + a_{3,2} + a_{3,3}) \cdot (b_{2,2} + b_{3,1} - b_{3,2})$$
$$M_{13} = (a_{1,3} + a_{3,3}) \cdot (-b_{2,2} + b_{3,2})$$
$$M_{14} = a_{1,3} \cdot b_{3,1}$$
$$M_{15} = (-a_{3,2} - a_{3,3}) \cdot (-b_{3,1} + b_{3,2})$$
$$M_{16} = (a_{1,3} + a_{2,2} - a_{2,3}) \cdot (b_{2,3} - b_{3,1} + b_{3,3})$$
$$M_{17} = (-a_{1,3} + a_{2,3}) \cdot (b_{2,3} + b_{3,3})$$
$$M_{18} = (a_{2,2} - a_{2,3}) \cdot (b_{3,1} - b_{3,3})$$

Laderman's scheme from 1976

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

where ...

$$M_{19} = a_{1,2} \cdot b_{2,1}$$
$$M_{20} = a_{2,3} \cdot b_{3,2}$$
$$M_{21} = a_{2,1} \cdot b_{1,3}$$
$$M_{22} = a_{3,1} \cdot b_{1,2}$$
$$M_{23} = a_{3,3} \cdot b_{3,3}$$

Other schemes with 23 multiplications

- While Strassen's scheme is essentially the only way to do the 2×2 case with 7 multiplications, there are several distinct schemes for 3×3 matrices using 23 multiplications.

Other schemes with 23 multiplications

- While Strassen's scheme is essentially the only way to do the 2×2 case with 7 multiplications, there are several distinct schemes for 3×3 matrices using 23 multiplications.
- If we insist in integer coefficients, there have so far (and to our knowledge) been only three other schemes for 3×3 matrices and 23 multiplications.

Other schemes with 23 multiplications

- While Strassen's scheme is essentially the only way to do the 2×2 case with 7 multiplications, there are several distinct schemes for 3×3 matrices using 23 multiplications.
- If we insist in integer coefficients, there have so far (and to our knowledge) been only three other schemes for 3×3 matrices and 23 multiplications.
- Using altogether about 35 years of computation time, we found more than 13000 new schemes for 3×3 and 23, and we expect that there are many others.

Other schemes with 23 multiplications

- While Strassen's scheme is essentially the only way to do the 2×2 case with 7 multiplications, there are several distinct schemes for 3×3 matrices using 23 multiplications.
- If we insist in integer coefficients, there have so far (and to our knowledge) been only three other schemes for 3×3 matrices and 23 multiplications.
- Using altogether about 35 years of computation time, we found more than 13000 new schemes for 3×3 and 23, and we expect that there are many others.
- Unfortunately we found no scheme with only 22 multiplications

How to Search for a Matrix Multiplication Scheme? (1)

$$
\begin{aligned}
M_1 &= (\alpha_{1,1}^{(1)} a_{1,1} + \alpha_{1,2}^{(1)} a_{1,2} + \cdots)(\beta_{1,1}^{(1)} b_{1,1} + \cdots) \\
M_2 &= (\alpha_{1,1}^{(2)} a_{1,1} + \alpha_{1,2}^{(2)} a_{1,2} + \cdots)(\beta_{1,1}^{(2)} b_{1,1} + \cdots) \\
&\vdots \\
c_{1,1} &= \gamma_{1,1}^{(1)} M_1 + \gamma_{1,1}^{(2)} M_2 + \cdots \\
&\vdots
\end{aligned}
$$

Set $c_{i,j} = \sum_k a_{i,k} b_{k,j}$ for all i, j and compare coefficients.

How to Search for a Matrix Multiplication Scheme? (2)

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i,j,k,l,m,n \in \{1,2,3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k}\delta_{i,m}\delta_{l,n}$$

The $\delta_{u,v}$ on the right refer to the Kronecker-delta, i.e., $\delta_{u,v} = 1$ if $u = v$ and $\delta_{u,v} = 0$ otherwise.

$3^6 = 729$ cubic equations

$23 \cdot 9 \cdot 3 = 621$ variables

How to Search for a Matrix Multiplication Scheme? (2)

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i,j,k,l,m,n \in \{1,2,3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k}\delta_{i,m}\delta_{l,n}$$

The $\delta_{u,v}$ on the right refer to the Kronecker-delta, i.e., $\delta_{u,v} = 1$ if $u = v$ and $\delta_{u,v} = 0$ otherwise.

$3^6 = 729$ cubic equations

$23 \cdot 9 \cdot 3 = 621$ variables

Laderman claims that he solved this system by hand, but he doesn't say exactly how.

How to Search for a Matrix Multiplication Scheme? (3)

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

The search space of the 3×3 case is enormous, even if $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ are restricted to the values in $\{-1, 0, 1\}$

How to Search for a Matrix Multiplication Scheme? (3)

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

The search space of the 3×3 case is enormous, even if $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ are restricted to the values in $\{-1, 0, 1\}$

Solution: Solve this system in $\mathbb{Z}_2$.

Reading $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ as boolean variables and $+$ as XOR, the problem becomes a SAT problem.

Notice that solutions in $\mathbb{Z}_2$ may not be solutions in $\mathbb{Z}$

Lifting

Remember the Brent equations:

$$\forall\, i,j,k,l,m,n \in \{1,2,3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k}\delta_{i,m}\delta_{l,n}$$

- Suppose we know a solution in $\mathbb{Z}_2$.
- Assume it came from a solution in $\mathbb{Z}$ with coefficients in $\{-1, 0, +1\}$.
- Then each $0 \in \mathbb{Z}_2$ was $0 \in \mathbb{Z}$ and each $1 \in \mathbb{Z}_2$ was $-1 \in \mathbb{Z}$ or $+1 \in \mathbb{Z}$.
- Plug the 0s of the $\mathbb{Z}_2$-solution into the Brent equations.
- Solve the resulting equations.

Lifting

Remember the Brent equations:

$$\forall\, i,j,k,l,m,n \in \{1,2,3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k}\delta_{i,m}\delta_{l,n}$$

- Suppose we know a solution in $\mathbb{Z}_2$.
- Assume it came from a solution in $\mathbb{Z}$ with coefficients in $\{-1, 0, +1\}$.
- Then each $0 \in \mathbb{Z}_2$ was $0 \in \mathbb{Z}$ and each $1 \in \mathbb{Z}_2$ was $-1 \in \mathbb{Z}$ or $+1 \in \mathbb{Z}$.
- Plug the 0s of the $\mathbb{Z}_2$-solution into the Brent equations.
- Solve the resulting equations.

Can every $\mathbb{Z}_2$-solution be lifted to a $\mathbb{Z}$-solution in this way?

- No, and we found some which don't admit a lifting.
- But they are very rare. In almost all cases, the lifting succeeds.

How to Search for a Matrix Multiplication Scheme? (4)

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i,j,k,l,m,n \in \{1,2,3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k}\delta_{i,m}\delta_{l,n}$$

Another solution: Solve this system by restricting equations with a zero righthand side to zero or two.

Still treat $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ as boolean variables.

Notice that this restriction removes solutions, but it even works for Laderman.

How to Search for a Matrix Multiplication Scheme? (4)

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k}\delta_{i,m}\delta_{l,n}$$

Another solution: Solve this system by restricting equations with a zero righthand side to zero or two.

Still treat $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ as boolean variables.

Notice that this restriction removes solutions, but it even works for Laderman.

Important challenge: how to break the symmetries?

Most effective approach so far: sort the $\delta_{j,k}\delta_{i,m}\delta_{l,n} = 1$ terms

Neighborhood Search

Neighborhood Search Results

So what?

- Okay, so there are many more matrix multiplication methods for 3×3 matrices with 23 coefficient multiplications than previously known.

So what?

- Okay, so there are many more matrix multiplication methods for 3×3 matrices with 23 coefficient multiplications than previously known.
- In fact, we have shown that the dimension of the algebraic set defined by the Brent equation is much larger than was previously known.

So what?

- Okay, so there are many more matrix multiplication methods for 3×3 matrices with 23 coefficient multiplications than previously known.
- In fact, we have shown that the dimension of the algebraic set defined by the Brent equation is much larger than was previously known.
- But none of this has any immediate implications on the complexity of matrix multiplication, neither theoretically nor practically.

So what?

- Okay, so there are many more matrix multiplication methods for 3×3 matrices with 23 coefficient multiplications than previously known.
- In fact, we have shown that the dimension of the algebraic set defined by the Brent equation is much larger than was previously known.
- But none of this has any immediate implications on the complexity of matrix multiplication, neither theoretically nor practically.
- In particular, it remains open whether there is a multiplication method for 3×3 matrices with 22 coefficient multiplications. If you find one, let us know.

What's Next?

Scheme Database

Check out our website for browsing through the schemes and families we found:

http://www.algebra.uni-linz.ac.at/research/matrix-multiplication/

Local Search for Fast Matrix Multiplication

Marijn Heule, Manuel Kauers, and Martina Seidl

THE UNIVERSITY OF TEXAS AT AUSTIN

JOHANNES KEPLER UNIVERSITÄT LINZ

Starting at Carnegie Mellon University in August

SAT 2019 Conference, Lisbon July 9, 2019