A family of schemes for multiplying 3×3 matrices with 23 coefficient multiplications

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Let R be a ring and S be a subring of the centralizer of R. Let $x_1, \dots, x_{17} \in S$ be arbitrary, set $x_{i,j} = x_i x_j + 1$ for $i, j = 1, \dots, 17$ and

$$p_1 = x_{2,3} + x_3 \qquad p_2 = x_7\, x_{5,6} + x_5 \qquad p_3 = x_4\, x_{2,3} + x_2$$

$$p_4 = x_{14}\, x_{12,13} + x_{12} \qquad p_5 = x_{16}\, x_{10,15} + x_{10} \qquad p_6 = x_4\, x_{2,3} + x_{3,4} + x_2$$

$$p_7 = x_8\, x_{11}\, x_{5,6} + x_8\, x_9\, x_{5,6} - x_6\, x_{11} \qquad p_8 = x_7\, x_8\, x_9\, x_{5,6} + x_7\, x_8\, x_{11}\, x_{5,6} - x_{11}\, x_{6,7} + x_5\, x_8\, x_9 + x_5\, x_8\, x_{11}.$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in R^{3\times3}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in R^{3\times3}, \qquad \text{and} \qquad C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = AB.$$

Then the entries of C can be computed from the entries of A and B as follows:

$$m_1 = (a_{11} + x_1\, a_{12} + a_{13}) \times (b_{33})$$
$$m_2 = (a_{11} + a_{21} - a_{22}) \times (b_{21} - b_{23})$$
$$m_3 = (a_{11} + a_{21} + a_{31} + a_{33}) \times (b_{11} + b_{12} - b_{13})$$
$$m_4 = (a_{11} + a_{21} + a_{32}) \times (b_{12} - b_{21} + b_{23})$$
$$m_5 = (a_{11} + a_{21}) \times (b_{11} - b_{13} + b_{21} - b_{23})$$
$$m_6 = (a_{11} - a_{23}) \times (b_{11} - b_{33})$$
$$m_7 = (a_{11} - a_{31} + a_{32}) \times (-b_{12})$$
$$m_8 = (a_{11} + a_{33}) \times (b_{12} - b_{13} + b_{33})$$
$$m_9 = (a_{11}) \times (b_{13} - b_{33})$$
$$m_{10} = (a_{12} + a_{13} + a_{22} + a_{23} + a_{32}) \times (b_{21} + b_{22} - b_{23})$$
$$m_{11} = (a_{12} + a_{13} + a_{23}) \times (b_{21} + b_{22} - b_{23} - b_{32})$$
$$m_{12} = (a_{12} + a_{13} - a_{33}) \times (b_{32})$$
$$m_{13} = (a_{12}) \times (x_{3,4}\, b_{22} - p_6\, b_{23} - x_{3,4}\, b_{32} + p_6\, x_1\, b_{33})$$
$$m_{14} = (a_{12}) \times (x_3\, b_{22} - p_1\, b_{23} - x_3\, b_{32} + p_1\, x_1\, b_{33})$$
$$m_{15} = (a_{13} + a_{23}) \times (-b_{21} - b_{22} + b_{23} + b_{31} + b_{32} - b_{33})$$
$$m_{16} = (-p_2\, a_{21} + p_2\, x_9\, a_{22} - p_2\, a_{23} - x_{6,7}\, a_{31} + p_8\, a_{32} - x_{6,7}\, a_{33}) \times (b_{11})$$
$$m_{17} = (x_{5,6}\, a_{21} - x_9\, x_{5,6}\, a_{22} + x_{5,6}\, a_{23} + x_6\, a_{31} - p_7\, a_{32} + x_6\, a_{33}) \times (b_{11})$$
$$m_{18} = (a_{22} + x_8\, a_{32}) \times (x_9\, b_{11} + b_{21})$$
$$m_{19} = (x_{15,16}\, a_{23} + p_5\, a_{33}) \times (b_{11} - b_{31})$$
$$m_{20} = (a_{32}) \times (x_{11}\, x_{13,14}\, b_{11} + p_4\, b_{12} - x_{13,14}\, b_{21} + p_4\, b_{22})$$
$$m_{21} = (-a_{32}) \times (x_{11}\, x_{13}\, b_{11} + x_{12,13}\, b_{12} - x_{13}\, b_{21} + x_{12,13}\, b_{22})$$
$$m_{22} = (x_{15}\, a_{23} + x_{10,15}\, a_{33}) \times (-b_{11} + b_{31})$$
$$m_{23} = (a_{33}) \times (x_{17}\, b_{11} - b_{12} + b_{13} - x_{17}\, b_{31} + b_{32} - b_{33})$$

$$c_{11} = m_1 + m_6 + m_{11} - x_{2,3}\, m_{13} + p_3\, m_{14} + m_{15} + x_{10,15}\, m_{19} + p_5\, m_{22}$$
$$c_{12} = m_8 + m_9 + m_{12} + p_1\, m_{13} - p_6\, m_{14} + x_{15}\, x_{17}\, m_{19} + x_{17}\, x_{15,16}\, m_{22} + m_{23}$$
$$c_{13} = m_1 + m_9 + x_3\, m_{13} - x_{3,4}\, m_{14}$$
$$c_{21} = x_6\, m_{16} + x_{6,7}\, m_{17} + m_{18} - x_{10,15}\, m_{19} + x_8\, x_{12,13}\, m_{20} + p_4\, x_8\, m_{21} - p_5\, m_{22}$$
$$c_{22} = m_2 + m_4 - m_8 - m_9 + m_{10} - m_{11} - m_{12} - x_{15}\, x_{17}\, m_{19} + x_{13}\, m_{20} + x_{13,14}\, m_{21} - x_{17}\, x_{15,16}\, m_{22} - m_{23}$$
$$c_{23} = m_2 - m_5 + m_6 - m_9 + x_6\, m_{16} + x_{6,7}\, m_{17} + m_{18} + x_8\, x_{12,13}\, m_{20} + p_4\, x_8\, m_{21}$$
$$c_{31} = -x_{5,6}\, m_{16} - p_2\, m_{17} + x_{15}\, m_{19} - x_{12,13}\, m_{20} - p_4\, m_{21} + x_{15,16}\, m_{22}$$
$$c_{32} = m_7 + m_8 + m_9 + x_{15}\, x_{17}\, m_{19} - x_{13}\, m_{20} - x_{13,14}\, m_{21} + x_{17}\, x_{15,16}\, m_{22} + m_{23}$$
$$c_{33} = -m_3 + m_4 + m_5 + m_7 + m_8 + m_9 - x_{5,6}\, m_{16} - p_2\, m_{17} - x_{12,13}\, m_{20} - p_4\, m_{21}$$

- It is easy (but tedious) to confirm the correctness of the above scheme by expanding all definitions and observing that we have $c_{i,j} = \sum_k a_{i,k} b_{k,j}$ for all i, j.
- The scheme performs only 23 multiplications of two elements of R, one for each m_k (plus some additions and some multiplications of elements of S with elements of R).
- Other schemes with 23 multiplications are known since 1976 [5], but no scheme with only 22 multiplications is known. To beat Strassen [8], we would need a scheme with 21.
- There is no way to instantiate the parameters $x_1, \dots, x_{17}$ such that the scheme can be simplified to a scheme with only 22 multiplications in R.
- The polynomials in $x_1, \dots, x_{17}$ appearing in the scheme describe variety of dimension 17. In this sense, there is no redundancy among the parameters.
- Compared to the families of Johnson and McLoughlin [4], our scheme has more parameters, and it requires no assumption on the coefficient ring R.
- Our family is unrelated to the family of [4] and to other known schemes [5, 1, 6, 7] for multiplying 3×3 matrices with 23 coefficient multiplications.
- Our scheme was found by a combination of SAT solving, described in more detail in [2], and computer algebra methods, described in more detail in [3].
- Some more schemes with 17 parameters, dozens with fewer parameters, and thousands of new isolated solutions are available here:

References

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