Conflict-Driven Clause Learning

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http://www.cs.cmu.edu/~mheule/15816-f22/

Automated Reasoning and Satisfiability

September 14, 2022

The Satisfiability (SAT) problem

$$
\begin{aligned}
&(x_5 \vee x_8 \vee \overline{x}_2) \wedge (x_2 \vee \overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_8 \vee \overline{x}_3 \vee \overline{x}_7) \wedge (\overline{x}_5 \vee x_3 \vee x_8) \wedge \\
&(\overline{x}_6 \vee \overline{x}_1 \vee \overline{x}_5) \wedge (x_8 \vee \overline{x}_9 \vee x_3) \wedge (x_2 \vee x_1 \vee x_3) \wedge (\overline{x}_1 \vee x_8 \vee x_4) \wedge \\
&(\overline{x}_9 \vee \overline{x}_6 \vee x_8) \wedge (x_8 \vee x_3 \vee \overline{x}_9) \wedge (x_9 \vee \overline{x}_3 \vee x_8) \wedge (x_6 \vee \overline{x}_9 \vee x_5) \wedge \\
&(x_2 \vee \overline{x}_3 \vee \overline{x}_8) \wedge (x_8 \vee \overline{x}_6 \vee \overline{x}_3) \wedge (x_8 \vee \overline{x}_3 \vee \overline{x}_1) \wedge (\overline{x}_8 \vee x_6 \vee \overline{x}_2) \wedge \\
&(x_7 \vee x_9 \vee \overline{x}_2) \wedge (x_8 \vee \overline{x}_9 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_9 \vee x_4) \wedge (x_8 \vee x_1 \vee \overline{x}_2) \wedge \\
&(x_3 \vee \overline{x}_4 \vee \overline{x}_6) \wedge (\overline{x}_1 \vee \overline{x}_7 \vee x_5) \wedge (\overline{x}_7 \vee x_1 \vee x_6) \wedge (\overline{x}_5 \vee x_4 \vee \overline{x}_6) \wedge \\
&(\overline{x}_4 \vee x_9 \vee \overline{x}_8) \wedge (x_2 \vee x_9 \vee x_1) \wedge (x_5 \vee \overline{x}_7 \vee x_1) \wedge (\overline{x}_7 \vee \overline{x}_9 \vee \overline{x}_6) \wedge \\
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&(x_6 \vee x_7 \vee \overline{x}_3) \wedge (\overline{x}_8 \vee \overline{x}_6 \vee \overline{x}_7) \wedge (x_6 \vee x_2 \vee x_3) \wedge (\overline{x}_8 \vee x_2 \vee x_5)
\end{aligned}
$$

Does there exist an assignment satisfying all clauses?

Search for a satisfying assignment (or proof none exists)

$$
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SAT Solver Paradigms Overview

DPLL: Aims at finding a small search-tree by selecting effective splitting variables (e.g. via looking ahead).

Strength: Effective on small, hard formulas.

Weakness: Expensive.

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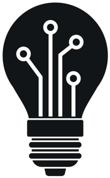
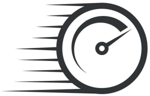


Local search: Given a full assignment for a formula Γ, flip the truth values of variables until satisfying Γ.

Strength: Can quickly find solutions for hard formulas.

Weakness: Cannot prove unsatisfiability.



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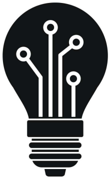

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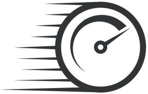

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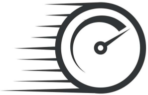

Conflict-driven clause learning (CDCL): Makes fast decisions and converts conflicts into learned clauses.

Strength: Effective on large, “easy” formulas.

Weakness: Hard to parallelize.

Conflict-driven Clause Learning: Overview

- Most successful architecture
- Superior on industrial benchmarks
- Brute-force?
  - Addition conflict clauses
  - Fast unit propagation
- Complete local search (for a refutation)?
- State-of-the-art (sequential) CDCL solvers:
  Kissat, CaDiCaL, Glucose, CryptoMiniSAT

Conflict-driven SAT solvers: Search and Analysis

$$
\begin{aligned}
&(x_1 \vee x_4) \wedge \\
&(x_3 \vee \overline{x}_4 \vee \overline{x}_5) \wedge \\
&(\overline{x}_3 \vee \overline{x}_2 \vee \overline{x}_4) \wedge \\
&\mathcal{F}_{\text{extra}}
\end{aligned}
$$

0

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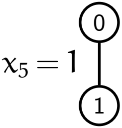

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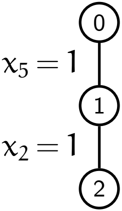

Conflict-driven SAT solvers: Search and Analysis

$$
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&(x_1 \lor x_4) \land \\
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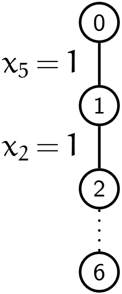

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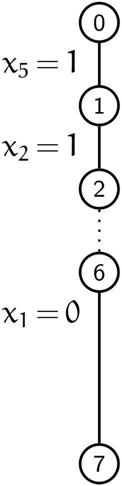

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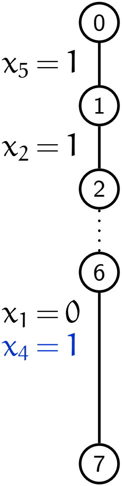

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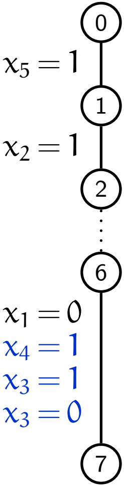

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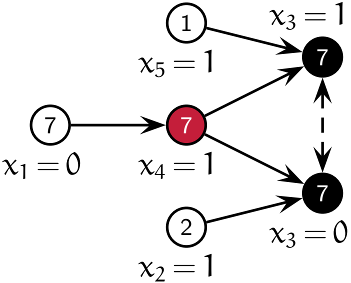

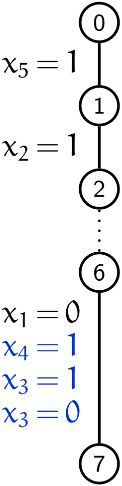

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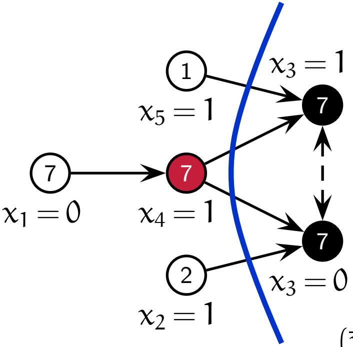

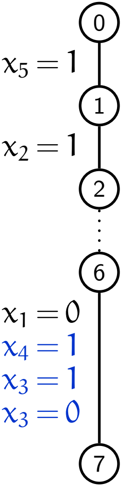

$(\overline{x}_2 \vee \overline{x}_4 \vee \overline{x}_5)$

Conflict-driven SAT solvers: Search and Analysis

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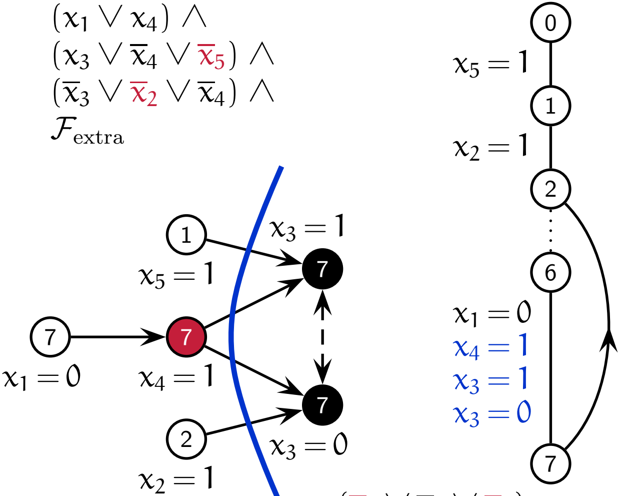

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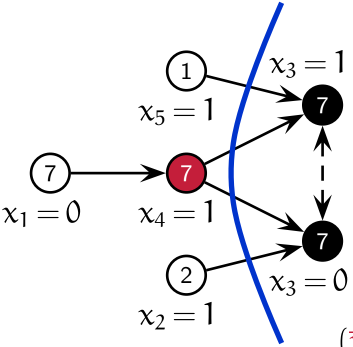

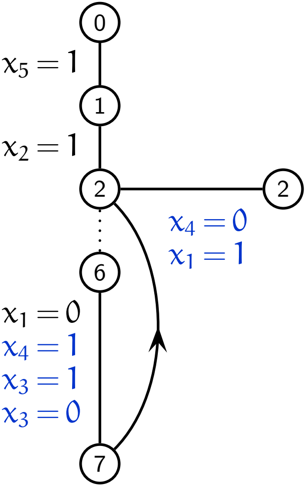

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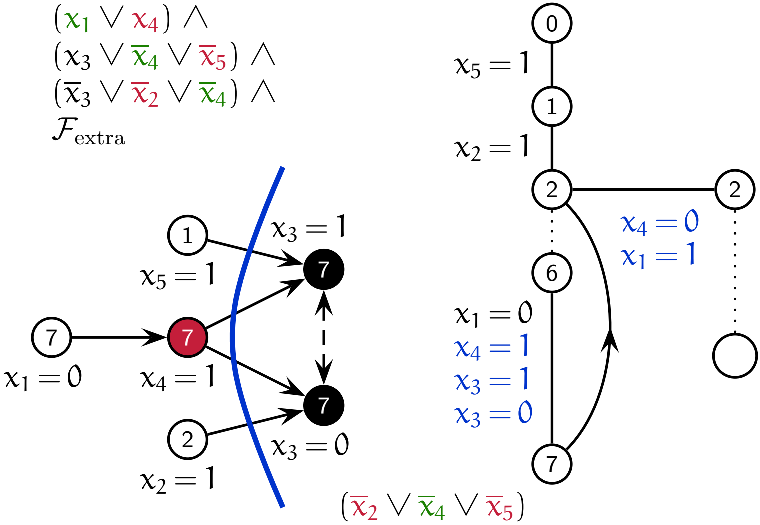

Implication graph [Marques-SilvaSakallah '96]

CDCL in a nutshell:

1. Main loop combines efficient problem simplification with cheap, but effective decision heuristics; ($> 90\%$ of time)
2. Reasoning kicks in if the current state is conflicting;
3. The current state is analyzed and turned into a constraint;
4. The constraint is added to the problem, the heuristics are updated, and the algorithm (partially) restarts.

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However, it has three weaknesses:

- CDCL is notoriously hard to parallelize;
- the representation impacts CDCL performance; and
- CDCL has exponential runtime on some "simple" problems.

Conflict-driven Clause Learning: Pseudo-code

while TRUE **do**

    l_{decision} := Decide ()

    If no l_{decision} **then return** `satisfiable`

    $\mathcal{F}$:= Simplify $(\mathcal{F}(l_{\text{decision}} \leftarrow 1))$

    while $\mathcal{F}$ contains $C_{\text{falsified}}$ **do**

        C_{conflict} := Analyze $(C_{\text{falsified}})$

        If $C_{\text{conflict}} = \emptyset$ **then return** `unsatisfiable`

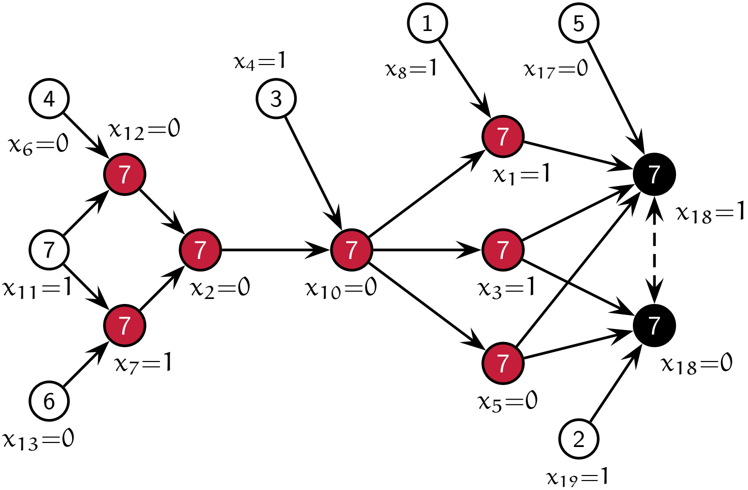
        BackTrack (C_{conflict})

        $\mathcal{F}$:= Simplify $(\mathcal{F} \cup \{C_{\text{conflict}}\})$

    end while

end while

Learning conflict clauses [Marques-SilvaSakallah'96]



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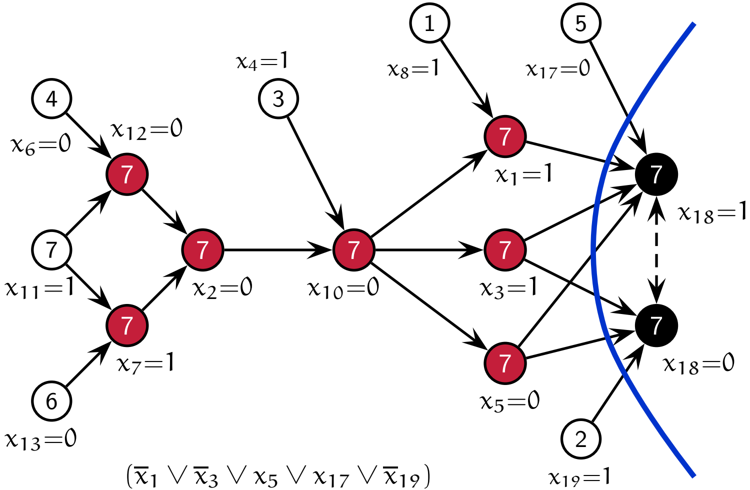

$(\overline{x}_1 \vee \overline{x}_3 \vee x_5 \vee x_{17} \vee \overline{x}_{19})$

tri-asserting clause

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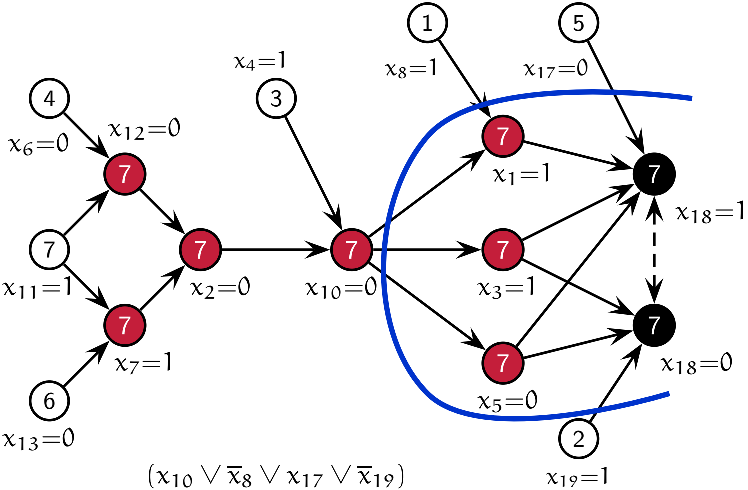

$(x_{10} \vee \overline{x}_8 \vee x_{17} \vee \overline{x}_{19})$

first unique implication point

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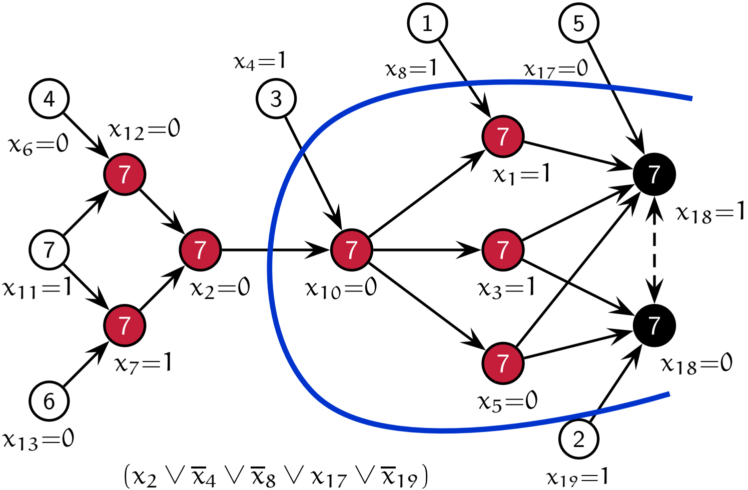
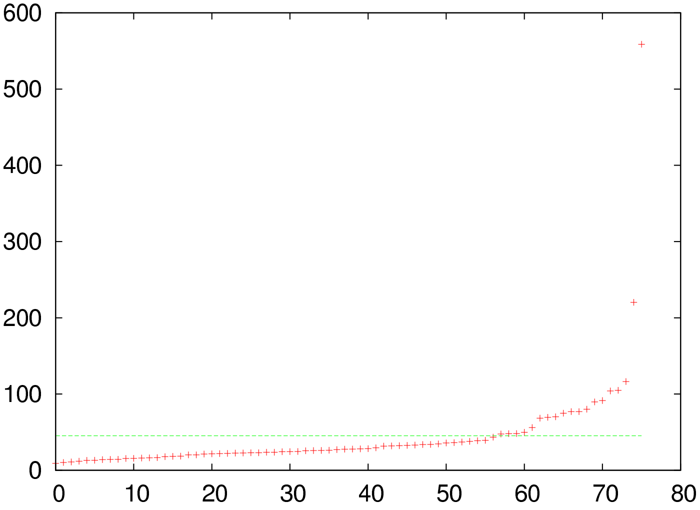


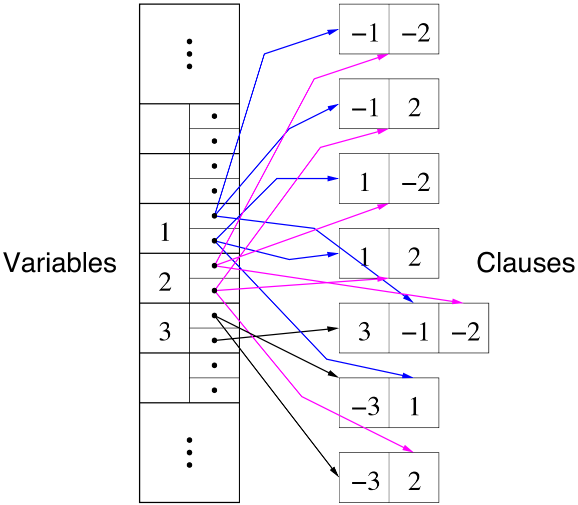
$(x_2 \vee \overline{x}_4 \vee \overline{x}_8 \vee x_{17} \vee \overline{x}_{19})$

second unique implication point

Average Learned Clause Length



Simple data structure for unit propagation



Conflict-driven: Watch pointers (1) [MoskewiczMZZM'01]

$$\varphi = \{x_1 = *, x_2 = *, x_3 = *, x_4 = *, x_5 = *, x_6 = *\}$$

$\overline{x}_1$	x_2	$\overline{x}_3$	$\overline{x}_5$	x_6

x_1	$\overline{x}_3$	x_4	$\overline{x}_5$	$\overline{x}_6$

Conflict-driven: Watch pointers (1) [MoskewiczMZZM'01]

$$\varphi = \{x_1 = *, x_2 = *, x_3 = *, x_4 = *, x_5 = 1, x_6 = *\}$$

$\overline{x}_1$	x_2	$\overline{x}_3$	$\overline{x}_5$	x_6

x_1	$\overline{x}_3$	x_4	$\overline{x}_5$	$\overline{x}_6$

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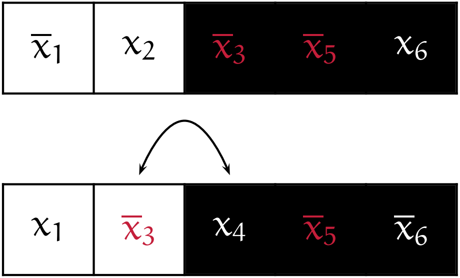

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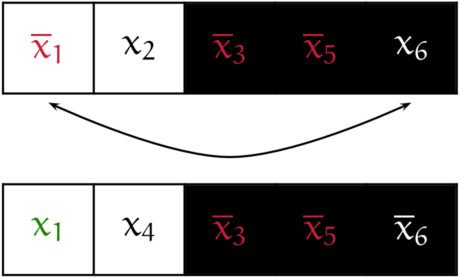

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x_1	x_4	$\overline{x}_3$	$\overline{x}_5$	$\overline{x}_6$

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$$\varphi = \{x_1 = 1, x_2 = *, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = *\}$$

x_6	x_2	$\overline{x}_3$	$\overline{x}_5$	$\overline{x}_1$

x_1	x_4	$\overline{x}_3$	$\overline{x}_5$	$\overline{x}_6$

Conflict-driven: Watch pointers (1) [MoskewiczMZZM'01]

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x_6	x_2	$\overline{x}_3$	$\overline{x}_5$	$\overline{x}_1$

x_1	x_4	$\overline{x}_3$	$\overline{x}_5$	$\overline{x}_6$

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Conflict-driven: Watch pointers (2) [MoskewiczMZZM'01]

Only examine (get in the cache) a clause when both

- a watch pointer gets falsified
- the other one is not satisfied

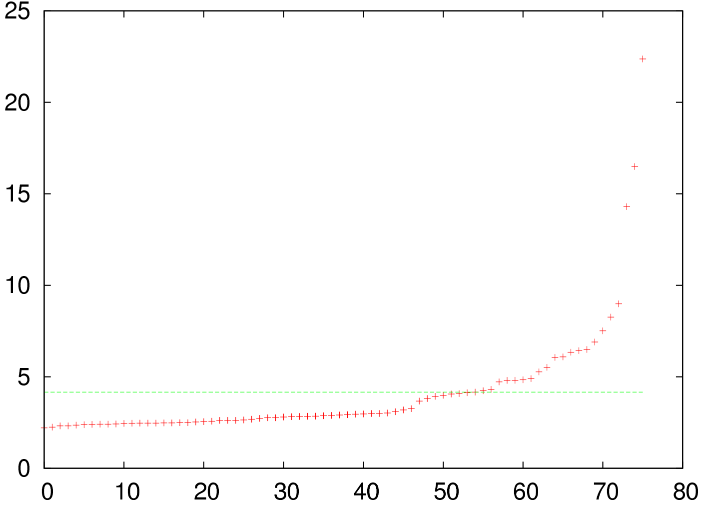
While backjumping, just unassign variables

Conflict clauses $\rightarrow$ watch pointers

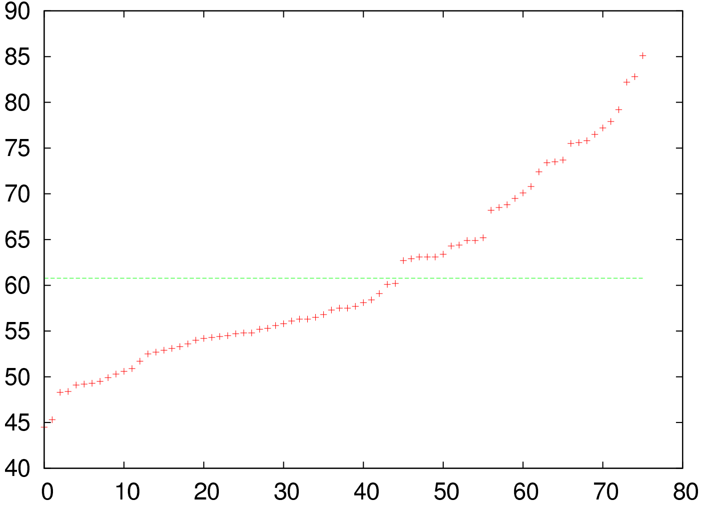
No detailed information available

Not used for binary clauses

Average Number Clauses Visited Per Propagation



Percentage visited clauses with other watched literal true



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Restart strategies

- aim: avoid heavy-tail behavior [GomesSelmanCrato'97]
- plus: focus search on recent conflicts when combined with dynamic heuristics

Variable selection heuristics

Based on the occurrences in the (reduced) formula

- examples: Jeroslow-Wang, Maximal Occurrence in clauses of Minimal Size (MOMS), look-aheads
- not practical for CDCL solver due to watch pointers

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Variable State Independent Decaying Sum (VSIDS)

- original idea (zChaff): for each conflict, increase the score of involved variables by 1, half all scores each 256 conflicts [MoskewiczMZZM'01]
- improvement (MiniSAT): for each conflict, increase the score of involved variables by δ and increase $\delta := 1.05\delta$ [EenSörensson'03]

Visualization of VSIDS in PicoSAT

`http://www.youtube.com/watch?v=MOjhFywLre8`

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Based on the encoding / consequently

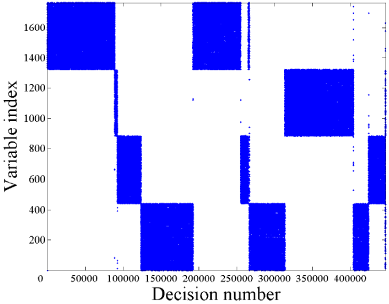
- negative branching (early MiniSAT) [EenSörensson'03]

Based on the last implied value (phase-saving)

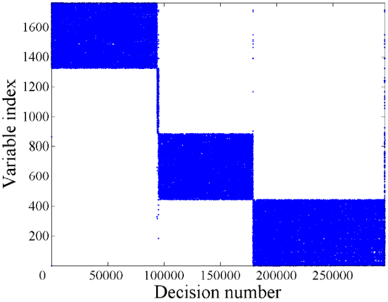
- introduced to CDCL [PipatsrisawatDarwiche'07]
- already used in local search [HirschKojevnikov'01]

Heuristics: Phase-saving [PipatsrisawatDarwiche'07]

Selecting the last implied value remembers solved components



negative branching



phase-saving

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Restart strategies: [Walsh'99, LubySinclairZuckerman'93]

- Geometrical restart: e.g. $100, 150, 225, 333, 500, 750, \ldots$
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Rapid restarts by reusing trail: [vanderTakHeuleRamos'11]

- Partial restart same effect as full restart
- Optimal strategy Luby-1: $1, 1, 2, 1, 1, 2, 4, \ldots$

Heuristics: SAT vs UNSAT [Oh'15]

The best heuristics choices depend on satisfiability: E.g.

- Restart frequently for UNSAT instances to get conflict early
- Restart sporadically for SAT instances to keep "progress"

Also, keeping learned clauses is less important on SAT instances and can actually slow down the search.

State-of-the-art CDCL solvers, such as CaDiCaL, have separate modes for SAT and UNSAT and they alternate between them.

Clause delection [EenSörensson'03, AudemardSimon'09]

Conflict clauses can significantly slow down CDCL solvers:

- Conflict clauses can quickly outnumber the original clauses
- Conflict clauses consists of important variables

Clause deletion is used to reduce the overhead:

- When the learned clause reach a limit, remove half
- Increase limit after every removal (completeness)

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Clause deletion heuristics:

- length of the clause
- relevance of the clause (when was it used in Analyze)
- the number of involved decision levels

Self-Subsumption

Use self-subsumption to shorten conflict clauses

$$\frac{C \vee l \quad D \vee \bar{l}}{D}\; C \subseteq D \qquad \frac{(a \vee b \vee l) \quad (a \vee b \vee c \vee \bar{l})}{(a \vee b \vee c)}$$

Conflict clause minimization is an important optimization.

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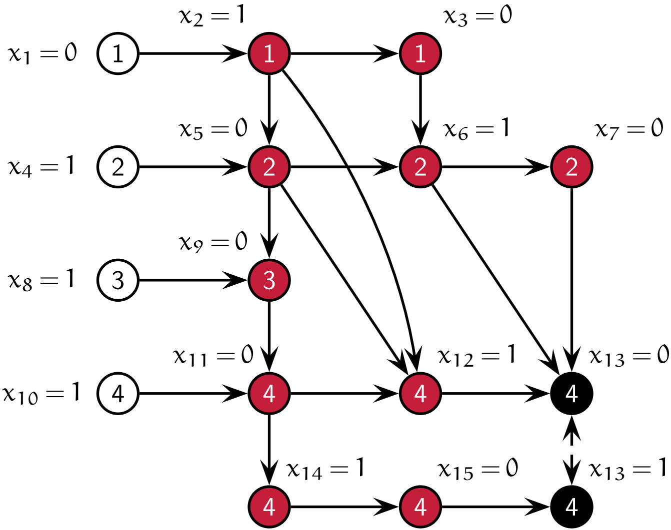
Conflict clause minimization is an important optimization.

Use implication chains to further minimization:

$$\dots(\bar{a} \vee b)(\bar{b} \vee c)(a \vee c \vee d)\dots \quad \Rightarrow$$
$$\dots(\bar{a} \vee b)(\bar{b} \vee c)(c \vee d)\dots$$

Conflict-clause minimization [SörenssonBiere'09]



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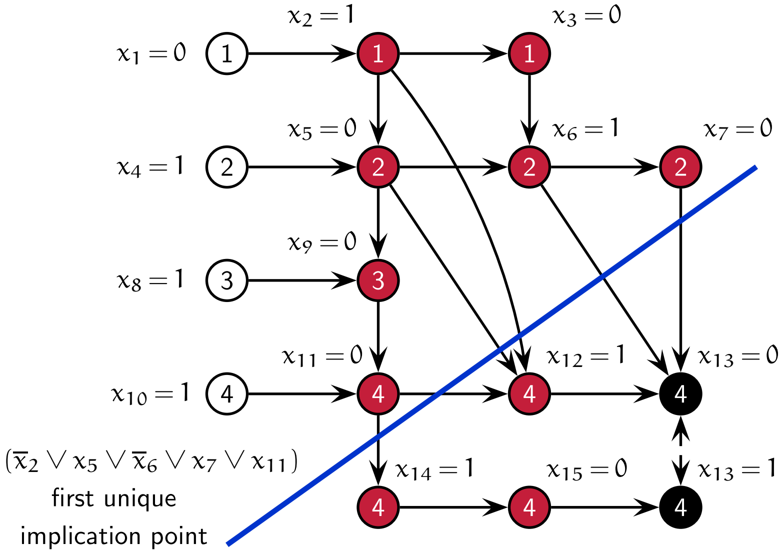

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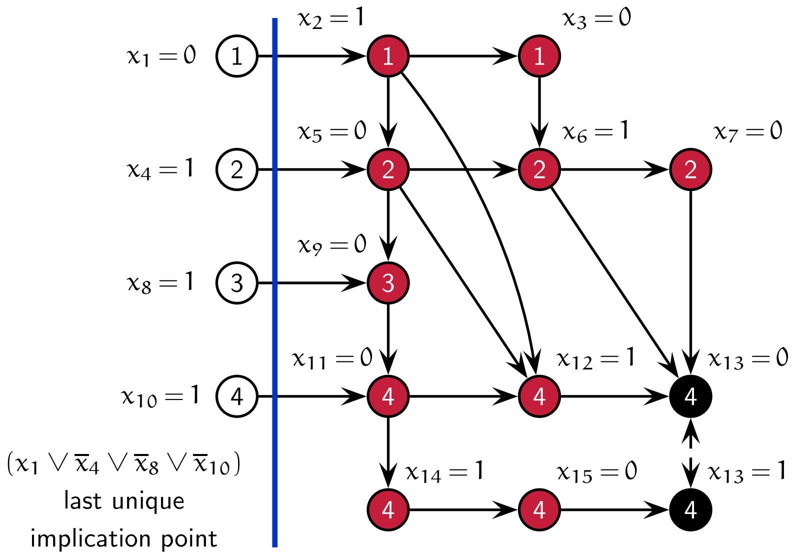

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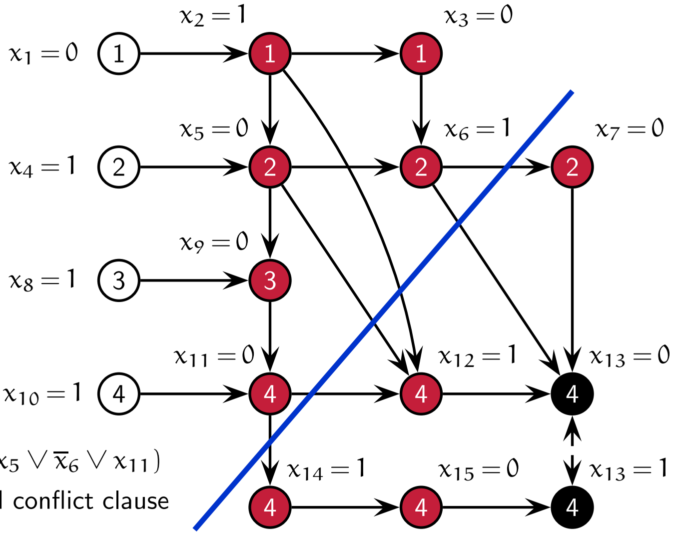
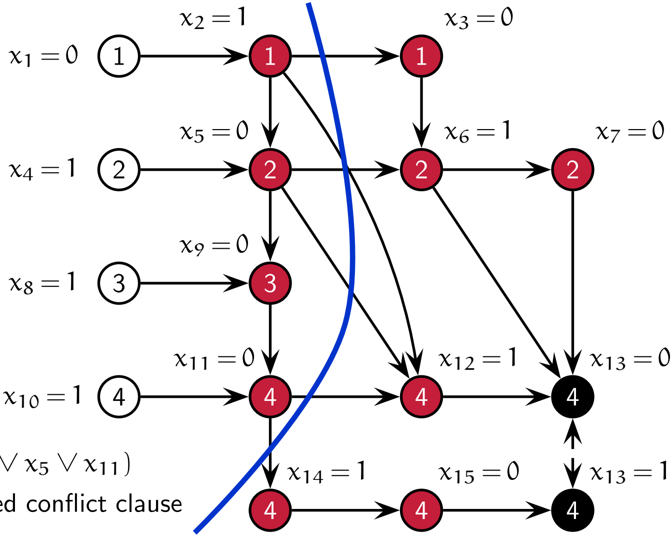


$(\overline{x}_2 \vee x_5 \vee \overline{x}_6 \vee x_{11})$

reduced conflict clause

Conflict-clause minimization [SörenssonBiere'09]



$(\overline{x}_2 \vee x_5 \vee x_{11})$

minimized conflict clause

Recent Advances

A new idea contributes to winning the competition.

Winner 2017: Clause vivification during search
[LuoLiXiaoManyáLü'17]

Winner 2018: Chronological backtracking
[NadelRyvchin'18]

Winner 2019: Multiple learnt clauses per conflict
[KochemazovZaikinKondratievSemenov'19]

Winner 2020: Back to C and "target phases"
[BiereFleury'20]

Conclusions: state-of-the-art CDCL solver

Key contributions to CDCL solvers:

- concept of conflict clauses (grasp) [Marques-SilvaSakallah'96]
- restart strategies [GomesSC'97,LubySZ'93]
- 2-watch pointers and VSIDS (zChaff) [MoskewiczMZZM'01]
- efficient implementation (Minisat) [EenSörensson'03]
- phase-saving (Rsat) [PipatsrisawatDarwiche'07]
- conflict-clause minimization [SörenssonBiere'09]
- SAT vs UNSAT [Oh'15]

+ Pre- and in-processing techniques