

Representations for Automated Reasoning

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Automated Reasoning and Satisfiability

September 15, 2020

Basic Constraints

Solver Input

Representing Integers

Cardinality Constraints

Lazy Encodings

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AtLeastOne

Given a set of Boolean variables x_1, \dots, x_n , how to encode

$$\text{ATLEASTONE}(x_1, \dots, x_n)$$

into SAT?

Hint: This is easy...

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$$(x_1 \vee x_2 \vee \dots \vee x_n)$$

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$$(x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$$

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Question: How many solutions does this formula have?

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Can we encode large XORs with **less clauses**?

Make it compact: $\text{XOR}(x_1, x_2, y) \wedge \text{XOR}(\bar{y}, x_3, \dots, x_n)$

Tradeoff: increase the number of variables but decreases the number of clauses!

AtMostOne: Pairwise Encoding

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Is it possible to use fewer clauses?

AtMostOne: Linear Encoding

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By splitting the constraint using additional variables. Apply the direct encoding if $n \leq 4$ otherwise replace $\text{ATMOSTONE}(x_1, \dots, x_n)$ by

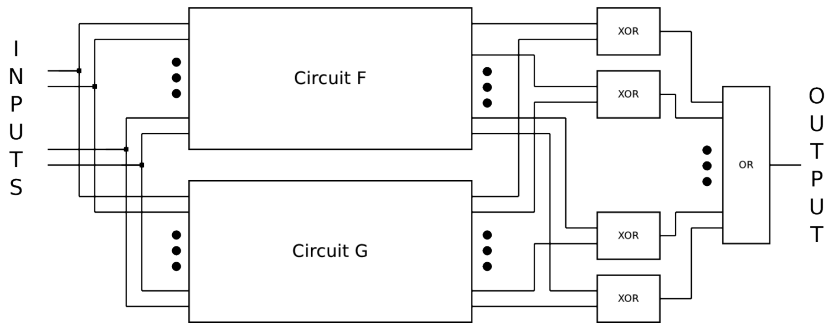
$$\text{ATMOSTONE}(x_1, x_2, x_3, y) \wedge \text{ATMOSTONE}(\bar{y}, x_4, \dots, x_n)$$

resulting in $3n - 6$ clauses and $(n - 3)/2$ new variables

AtMostOne: Equivalence

How to show that two encodings of $\text{ATMOSTONE}(x_1, x_2)$ are equivalent?

If we have a circuit representation of each encoding then we can use a **miter** circuit to show that for the same inputs, the output variables are equivalent:



AtMostOne: Equivalence

Are these two encoding of $\text{ATMOSTONE}(x_1, x_2)$ equivalent?

φ_1 (direct encoding)	φ_2 (split encoding)
$\bar{x}_1 \vee \bar{x}_2$	$\bar{x}_1 \vee y$
	$\bar{y} \vee \bar{x}_2$

Question: Is φ_1 equivalent to φ_2 ?

Note: $\varphi_1 \leftrightarrow \varphi_2$ is **valid** if $\neg\varphi_1 \wedge \varphi_2$ and $\varphi_1 \wedge \neg\varphi_2$ are **unsatisfiable**.

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Note: $\neg\varphi_1 \equiv x_1 \wedge x_2$

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φ_1 and φ_2 are **equisatisfiable**:

- ▶ φ_1 is satisfiable iff φ_2 is satisfiable.

Note: Equisatisfiability is weaker than equivalence but useful if all we want we want to do is determine satisfiability.

Basic Constraints

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Solver Input: DIMACS format

c famous problem (in CNF)

p cnf 6 9

1 4 0

2 5 0

3 6 0

-1 -2 0

-1 -3 0

-2 -3 0

-4 -5 0

-4 -6 0

-5 -6 0

Solver Input: DIMACS format

c pigeon hole problem

p cnf 6 9

```
1 4 0          #  pigeon[1]@hole[1] ∨  pigeon[1]@hole[2]
2 5 0          #  pigeon[2]@hole[1] ∨  pigeon[2]@hole[2]
3 6 0          #  pigeon[3]@hole[1] ∨  pigeon[3]@hole[2]
-1 -2 0        #  ¬pigeon[1]@hole[1] ∨ ¬pigeon[2]@hole[1]
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- ▶ SAT solvers take as input a formula in CNF
- ▶ What is the complexity of transformation any formula φ in CNF?

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In some cases, converting a formula to CNF can have an **exponential** explosion on the size of the formula.

If we convert $(x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \dots \vee (x_n \wedge y_n)$ using De Morgan's laws and distributive law to CNF:

$$(x_1 \vee x_2 \vee \dots \vee x_n) \wedge (y_1 \vee x_2 \vee \dots \vee x_n) \wedge \dots \wedge (y_1 \vee y_2 \vee \dots \vee y_n)$$

- ▶ How can we avoid the exponential blowup? In this case, the equivalent formula would have 2^n clauses!

Solver Input: Tseitin Transformation (1)

- ▶ SAT solvers take as input a formula in CNF
- ▶ What is the complexity of transformation any formula φ in CNF?

- ▶ Tseitin's transformation converts a formula φ into an **equisatisfiable** CNF formula that is linear in the size of φ !
- ▶ **Key idea:** introduce auxiliary variables to represent the output of subformulas, and constrain those variables using CNF clauses!

Solver Input: Tseitin Transformation (2)

$$p \rightarrow (q \wedge r)$$

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$$\begin{aligned}t_1 &\leftrightarrow p \rightarrow t_2 \\t_2 &\leftrightarrow q \wedge r\end{aligned}$$

Solver Input: Tseitin Transformation (2)

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1. Introduce a fresh variable for every non-atomic subformula
2. Convert each equivalence into CNF

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1. Introduce a fresh variable for every non-atomic subformula
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3. Assert the conjunction of t_1 and the CNF-converted equivalences

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$$t_1 \wedge F_1 \wedge F_2$$

Solver Input: Tseitin Transformation (3)

- ▶ Using automated tools to encode to CNF:
limboole: <http://fmv.jku.at/limboole>

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- ▶ Using automated tools to encode to CNF:
limboole: <http://fmv.jku.at/limboole>
- ▶ Tseitin's encoding may add many redundant variables/clauses!
- ▶ Using **limboole** for the pigeon hole problem ($n = 3$) creates a formula with 40 variables and 98 clauses
- ▶ After unit propagation the formula has 12 variables and 28 clauses
- ▶ Original CNF formula only has 6 variables and 9 clauses

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Representing Integers: Direct Encoding

- ▶ Each number i is represented by a Boolean variable: d_i
- ▶ At least one number is true: $d_0 \vee \dots \vee d_n$
- ▶ At most one number is true: $\bigwedge_{i < j} \bar{d}_i \vee \bar{d}_j$
- ▶ Expressing in a clause that an integer has a specific value v requires one literal.
- ▶ For example, “if the number is 1, then do x ”, is encoded as $\bar{d}_1 \vee x$.
- ▶ Typically effective when reasoning about a small range of integers.

Representing Integers: Order Encoding

Order encoding:

- ▶ Variables represent that a number is larger or equal: $o_{\geq i}$
- ▶ Requires a linear number of binary clauses: $o_{\geq i} \vee \bar{o}_{\geq i+1}$
- ▶ Expressing in a clause that an integer has a specific value v requires two literals.
- ▶ For example, “if the number is 1, then do x ”, is encoded as $\bar{o}_{\geq 1} \vee o_{\geq 2} \vee x$.
- ▶ Allows the solver to reason (and produce clauses) that cover multiple cases.

Representing Integers: Binary Encoding

Binary encoding:

- ▶ Use $\lceil \log_2 n \rceil$ auxiliary variables b_i to represent n in binary
- ▶ All non-occurring numbers $\leq \lceil \log_2 n \rceil$ need to be blocked. For example, if we have the numbers 0, 1, and 2, then the number 3 needs to be blocked: $(\neg b_0 \vee \neg b_1)$
- ▶ Expressing in a clause that an integer has a specific value v requires $\lceil \log_2 n \rceil$ literals.
- ▶ For example, “if the number is 1, then do x ”, is encoded as $\neg b_0 \vee b_1 \vee x$.
- ▶ Typically effective when reasoning about a large range of integers.

Basic Constraints

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How to encode cardinality constraints?

Recall `ATMOSTONE` constraints:

- ▶ Direct encoding for `ATMOSTONE` constraints:
- ▶ `ATMOSTONE`: $x_1 + x_2 + x_3 + x_4 \leq 1$
- ▶ Clauses:

$$\left. \begin{array}{l} (x_1 \rightarrow \bar{x}_2) \\ (x_1 \rightarrow \bar{x}_3) \\ (x_1 \rightarrow \bar{x}_4) \\ \dots \end{array} \right\} \begin{array}{l} \bar{x}_1 \vee \bar{x}_2 \\ \bar{x}_1 \vee \bar{x}_3 \\ \bar{x}_1 \vee \bar{x}_4 \\ \dots \end{array}$$

- ▶ Complexity: $\mathcal{O}(n^2)$ clauses

How to encode cardinality constraints?

ATMOSTK constraints:

- ▶ Naive encoding for ATMOSTK constraints:
- ▶ Cardinality constraint: $x_1 + x_2 + x_3 + x_4 \leq 2$
- ▶ Clauses:

$$\left. \begin{array}{l} (x_1 \wedge x_2 \rightarrow \bar{x}_3) \\ (x_1 \wedge x_2 \rightarrow \bar{x}_4) \\ (x_2 \wedge x_3 \rightarrow \bar{x}_4) \\ \dots \end{array} \right\} \begin{array}{l} (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \\ (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_4) \\ (\bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4) \\ \dots \end{array}$$

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- ▶ Complexity: $\mathcal{O}(n^k)$ clauses
- ▶ What **properties** should these encodings have?
Number of variables? Number of clauses? Other?

Consistency and Arc-Consistency (1)

- ▶ Let us consider an encoding of a constraint C such that there is a correspondence between assignments of the variables in C with Boolean assignments of the variables in the encoding
- ▶ The encoding is **consistent** if whenever M is partial assignment inconsistent wrt C (i.e., cannot be extended to a solution of C), unit propagation leads to conflict

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- ▶ The encoding is **consistent** if whenever M is partial assignment inconsistent wrt C (i.e., cannot be extended to a solution of C), unit propagation leads to conflict
- ▶ The encoding is **arc-consistent** if
 1. it is consistent, and
 2. unit propagation discards arc-inconsistent values (values that cannot be assigned)
- ▶ These are good properties for encodings: SAT solvers are very good at **unit propagation!**

Consistency and Arc-Consistency (2)

In the case of the `ATMOSTONE` constraint

$$x_1 + x_2 + \dots + x_n \leq 1:$$

- ▶ **Consistency** \equiv if there are two variables x_i assigned to *true* then unit propagation should give a conflict
- ▶ **Arc-consistency** \equiv Consistency + if there is one x_i assigned to *true* then all others x_j should be assigned to *false* by unit propagation

Cardinality Constraints: Sinz encoding (1)

Can we build an encoding that is arc-consistent and uses a polynomial number of variables/clauses for at-most-k constraints?

Cardinality Constraints: Sinz encoding (1)

Can we build an encoding that is arc-consistent and uses a polynomial number of variables/clauses for at-most-k constraints?

Yes! By adding $O(n \cdot k)$ auxiliary variables we only need $O(n \cdot k)$ clauses!

Cardinality Constraints: Sinz encoding (2)

$$x_1 + x_2 + x_3 \leq 2$$

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Note: this is easy to encode but we will use it to give intuition.
How would you encode this with a single clause?

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Note: this is easy to encode but we will use it to give intuition.

How would you encode this with a single clause?

$$\neg(x_1 \wedge x_2 \wedge x_3) \equiv (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

Cardinality Constraints: Sinz encoding (2)

$$x_1 + x_2 + x_3 \leq 2$$

x_1	x_2	x_3
$s_{1,1}$	$s_{2,1}$	$s_{3,1}$
—	$s_{2,2}$	$s_{3,2}$
—	—	$s_{3,3}$

- $s_{i,j} \equiv$ At least j variables x_1, \dots, x_i are assigned 1

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x_1	x_2	x_3
$s_{1,1}$	$s_{2,1}$	$s_{3,1}$
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► $x_1 \rightarrow ???$

Cardinality Constraints: Sinz encoding (2)

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x_1	x_2	x_3
$s_{1,1}$	$s_{2,1}$	$s_{3,1}$
—	$s_{2,2}$	$s_{3,2}$
—	—	$s_{3,3}$

▶ $x_1 \rightarrow s_{1,1}$

▶ $x_2 \rightarrow s_{2,1}$

▶ $x_3 \rightarrow s_{3,1}$

Cardinality Constraints: Sinz encoding (2)

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x_1	x_2	x_3
$s_{1,1}$	$s_{2,1}$	$s_{3,1}$
—	$s_{2,2}$	$s_{3,2}$
—	—	$s_{3,3}$

► $s_{1,1} \rightarrow ???$

Cardinality Constraints: Sinz encoding (2)

$$x_1 + x_2 + x_3 \leq 2$$

x_1	x_2	x_3
$s_{1,1}$	$s_{2,1}$	$s_{3,1}$
—	$s_{2,2}$	$s_{3,2}$
—	—	$s_{3,3}$

▶ $s_{1,1} \rightarrow s_{2,1}$

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—	$s_{2,2}$	$s_{3,2}$
—	—	$s_{3,3}$

► $(x_2 \wedge s_{1,1}) \rightarrow ???$

Cardinality Constraints: Sinz encoding (2)

$$x_1 + x_2 + x_3 \leq 2$$

x_1	x_2	x_3
$s_{1,1}$	$s_{2,1}$	$s_{3,1}$
—	$s_{2,2}$	$s_{3,2}$
—	—	$s_{3,3}$

▶ $(x_2 \wedge s_{1,1}) \rightarrow s_{2,2}$

▶ $(x_3 \wedge s_{2,1}) \rightarrow s_{3,2}$

▶ $(x_3 \wedge s_{2,2}) \rightarrow s_{3,3}$

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$s_{1,1}$	$s_{2,1}$	$s_{3,1}$
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—	—	$s_{3,3}$

- ▶ What are we missing?
- ▶ We need to enforce that at most two x_i are assigned to 1. How can we do this?

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—	—	$s_{3,3}$

- ▶ What are we missing?
- ▶ We need to enforce that at most two x_i are assigned to 1. How can we do this?
- ▶ $\bar{s}_{3,3}$

Cardinality Constraints: Sinz encoding (2)

$$x_1 + x_2 + x_3 \leq 2$$

p cnf 9 10

-1 4 0	# $\bar{x}_1 \vee s_{1,1}$
-2 5 0	# $\bar{x}_2 \vee s_{2,1}$
-3 7 0	# $\bar{x}_3 \vee s_{3,1}$
-4 5 0	# $\bar{s}_{1,2} \vee s_{2,1}$
-5 7 0	# $\bar{s}_{2,1} \vee s_{3,1}$
-6 8 0	# $\bar{s}_{2,2} \vee s_{3,2}$
-2 -4 6 0	# $\bar{x}_2 \vee \bar{s}_{1,1} \vee s_{2,2}$
-3 -5 8 0	# $\bar{x}_3 \vee \bar{s}_{2,1} \vee s_{3,2}$
-3 -6 9 0	# $\bar{x}_3 \vee \bar{s}_{2,2} \vee s_{3,3}$
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Cardinality Constraints: Sinz encoding (2)

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-9 0	# $\bar{s}_{3,3}$

If $x_1 = 1$ and $x_2 = 2$ then by unit propagation we have $x_3 = 0$.

Cardinality Constraints: Sinz encoding (2)

$$x_1 + x_2 + x_3 \leq 2$$

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-1 4 0	# $\bar{x}_1 \vee s_{1,1}$
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If $x_1 = 1$ and $x_2 = 2$ then by unit propagation we have $x_3 = 0$.

Cardinality Constraints: Sinz encoding (3)

Encoding for the general case $x_1 + \dots + x_n \leq k$:

$$\begin{aligned} & (\bar{x}_1 \vee s_{1,1}) \\ & (\bar{s}_{1,j}) \quad \text{for } 1 < j \leq k \\ & \left. \begin{aligned} & (\bar{x}_i \vee s_{i,1}) \\ & (\bar{s}_{i-1,1} \vee s_{i,1}) \\ & (\bar{s}_i \vee \bar{s}_{i-1,k}) \end{aligned} \right\} \quad \text{for } 1 < i < n \\ & \left. \begin{aligned} & (\bar{x}_i \vee \bar{s}_{i-1,j-1} \vee s_{i,j}) \\ & (\bar{s}_{i-1,j} \vee s_{i,j}) \end{aligned} \right\} \quad \text{for } 1 < i < n \text{ and } 1 < j \leq k \\ & (\bar{x}_n \vee \bar{s}_{n-1,k}) \end{aligned}$$

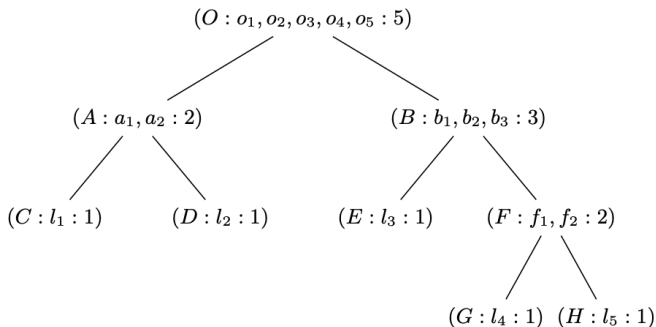
More details in paper: “Towards an Optimal CNF Encoding of Boolean Cardinality Constraints”, CP2005

- ▶ This version considers extra auxiliary variables that can be removed (e.g., sum at x_1 is never greater than 1)

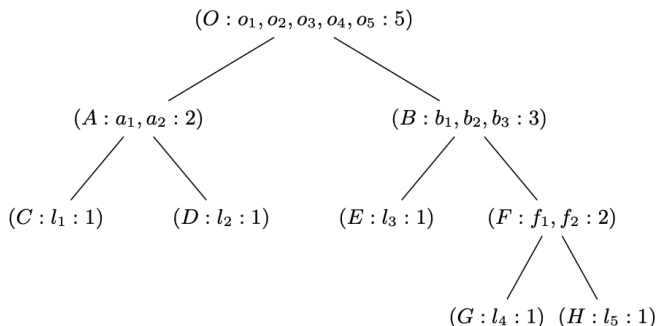
Cardinality Constraints: Totalizer encoding (1)

What is another example of an at-most-k encoding for $l_1 + \dots + l_5 \leq k$?

Totalizer encoding is based on a tree structure and also only needs $O(n \cdot k)$ clauses/variables.

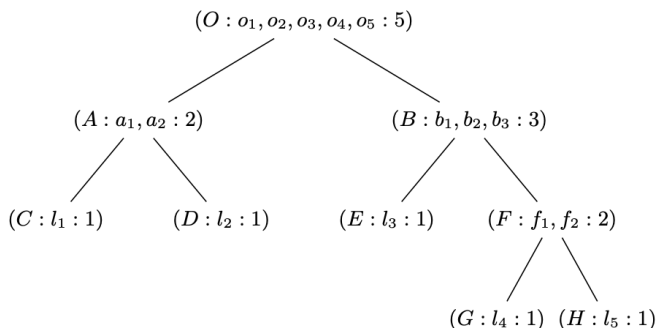


Cardinality Constraints: Totalizer encoding (2)



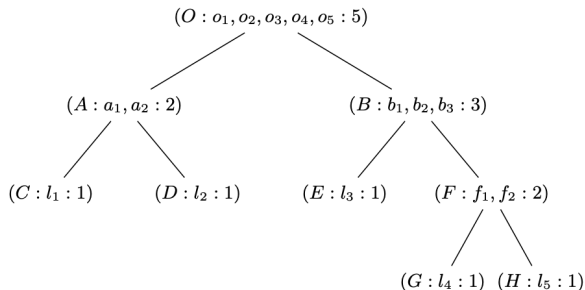
- ▶ Use auxiliary variables to count the sum of the subtree:
 - ▶ $f_1 \equiv l_4 + l_5 = 1$
 - ▶ $f_2 \equiv l_4 + l_5 = 2$
- ▶ Note that only f_1 or f_2 will be assigned to 1.

Cardinality Constraints: Totalizer encoding (3)



- ▶ Use auxiliary variables to count the sum of the subtree:
 - ▶ $b_1 \equiv l_3 + f_1 + 2 \times f_2 = 1$
 - ▶ $b_2 \equiv l_3 + f_1 + 2 \times f_2 = 2$
 - ▶ $b_3 \equiv l_3 + f_1 + 2 \times f_2 = 3$

Cardinality Constraints: Totalizer encoding (4)



Any parent node P , counting up to n_P , has two children L and R counting up to n_L and n_R respectively s.t. $n_L + n_R = n_P$.

Further reading

More details about cardinality encodings can be found in:

- ▶ Sinz's encoding:
Carsten Sinz. Towards an Optimal CNF Encoding of Boolean Cardinality Constraints. CP 2005. pp. 827-831
<http://www.carstensinz.de/papers/CP-2005.pdf>
- ▶ Totalizer encoding:
Olivier Bailleux, Yacine Boufkhad. Efficient CNF Encoding of Boolean Cardinality Constraints. CP 2003. pp. 108-122
<https://tinyurl.com/y6ph76au>
- ▶ Modulo Totalizer encoding:
Toru Ogawa, Yangyang Liu, Ryuzo Hasegawa, Miyuki Koshimura, Hiroshi Fujita. Modulo Based CNF Encoding of Cardinality Constraints and Its Application to MaxSAT Solvers. ICTAI 2013. pp. 9-17 <https://ieeexplore.ieee.org/document/6735224>
- ▶ Cardinality networks:
Roberto Asin, Robert Nieuwenhuis, Albert Oliveras, Enric Rodriguez-Carbonell. Cardinality Networks and Their Applications. SAT 2009. pp. 167-180 <https://tinyurl.com/yxwrzxxo>

Other encodings

Many other encodings exist for cardinality constraints!

Majority are based on circuits!

Example: Sorting Networks use $O(n \log^2 k)$ variables and clauses

We can also generalize to linear constraints with integer coefficients called **pseudo-Boolean** constraints:

$$a_1x_1 + \dots + a_nx_n \leq k$$

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Question: Can we generalize Sinz's encoding to pseudo-Boolean constraints?

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Question: Can we generalize Sinz's encoding to pseudo-Boolean constraints? **Yes!** We just need to consider the coefficient when writing the sum constraints.

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$$a_1 x_1 + \dots + a_n x_n \leq k$$

Question: Can we generalize Sinz's encoding to pseudo-Boolean constraints? **Yes!** We just need to consider the coefficient when writing the sum constraints.

More efficient encodings: **Binary merger** encoding only requires $O(n^2 \log^2(n) \log(w_{\max}))$ clauses and maintains arc-consistency!

Basic Constraints

Solver Input

Representing Integers

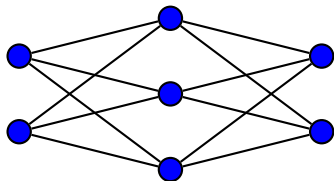
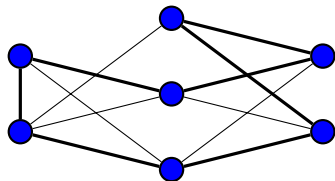
Cardinality Constraints

Lazy Encodings

Hamiltonian Cycles: Two Constraints

Hamiltonian Cycle Problem (HCP):

Does there exist a cycle that visits **all vertices exactly once**?



Two constraints:

- ▶ Exactly two edges per vertex: easy cardinality constraints
- ▶ Exactly one cycle: hard to be compact and arc-consistent
 - ▶ One option is to ignore the constraint: **incremental SAT**.
 - ▶ Various encodings use $O(|V|^3)$. **Too large** for many graphs.
 - ▶ For large graphs we need **quasi-linear encodings** in $|E|$.

Hamiltonian Cycles: Incremental SAT

Lazy encoding: instead of encoding the connectivity constraint eagerly, encode it lazily!

Every time the solver returns a solution:

1. Check if it is connected. If it is then we found a solution.
2. Otherwise, add constraints to force connectivity of the current path. Ask for a new solution [Go to 1].

In practice, we can find a solution without adding add subtours! Even though we need to perform several SAT calls to find the solution, this is often faster than most encodings into one large SAT formula.

Hamiltonian Cycles: Binary Adder Encoding [Zhou 2020]

The index of each vertex is a binary number. One vertex has index 0. If edge variable $e_{u,v}$ is assigned to true then the index of v is the successor of the index of u .

Example

Let $|V| = 7$, thus $k = \lceil \log_2 7 \rceil = 3$. For vertex v , variables v_2 , v_4 , and v_8 denote the least, middle, and most significant bit, respectively. For an edge variable $e_{u,v}$, we use the constraints:

$$\begin{aligned}e_{u,v} &\rightarrow (u_2 \leftrightarrow v_2) \\(e_{u,v} \wedge \bar{u}_2) &\rightarrow (u_4 \leftrightarrow v_4) \\(e_{u,v} \wedge u_2) &\rightarrow (u_4 \leftrightarrow v_4) \\(e_{u,v} \wedge \bar{u}_2) &\rightarrow (u_8 \leftrightarrow v_8) \\(e_{u,v} \wedge \bar{u}_4) &\rightarrow (u_8 \leftrightarrow v_8) \\(e_{u,v} \wedge u_2 \wedge u_4) &\rightarrow (u_8 \leftrightarrow v_8)\end{aligned}$$

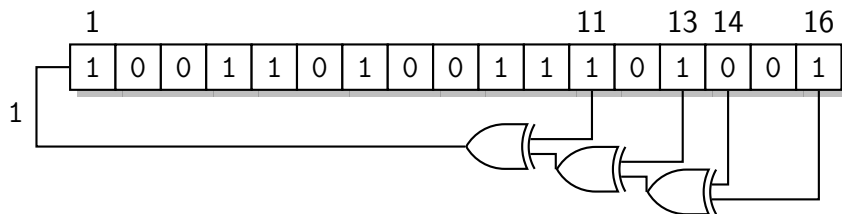
Hamiltonian Cycles: Linear-Feedback Shift Register

A k -bit **Linear-Feedback Shift Register** (LFSR) loops through $\{1, \dots, 2^k - 1\}$ by shifting all bits one position to the left and placing the parity of some bits in the vacated position

Example

An example LFSR of 16 bits is $1 + x^{11} + x^{13} + x^{14} + x^{16}$, which has $2^{16} - 1 = 65,535$ states. The figure below shows an illustration of this LFSR with state 10011010011101001.

The next state is 11001101001110100.



Hamiltonian Cycles: LFSR Encoding [Johnson 2018]

Enforcing the successor property using LFSR is compact and has been used to efficiently find Hamiltonian cycles in Erin and Stedman triples.

Example

Let $|V| = 7$, thus $k = \lceil \log_2 7 + 1 \rceil = 3$. We use 3-bit LFSR $1 + x + x^2$. The bit-vector variables of vertex v are $v_{7,1}$, $v_{7,2}$, and $v_{7,3}$. For an edge variable $e_{u,v}$, we add the constraints:

$$e_{u,v} \rightarrow (v_{7,1} \leftrightarrow (u_{7,2} \leftrightarrow u_{7,3}))$$

$$e_{u,v} \rightarrow (v_{7,2} \leftrightarrow u_{7,1})$$

$$e_{u,v} \rightarrow (v_{7,3} \leftrightarrow u_{7,2})$$

Hamiltonian Cycles: Chinese Remainder Encoding

Can we get the best all three worlds?

- ▶ Incremental SAT: Only partially encode the hard constraint
- ▶ Binary adder: refute some cycles quickly
- ▶ LSFR: few and short clauses, no auxiliary variables

Hamiltonian Cycles: Chinese Remainder Encoding

Can we get the best all three worlds?

- ▶ Incremental SAT: Only partially encode the hard constraint
- ▶ Binary adder: refute some cycles quickly
- ▶ LSFR: few and short clauses, no auxiliary variables

Chinese remainder encoding:

- ▶ Block all subcycles except one of length $0 \pmod{m}$
- ▶ Pick $m < |V|$ and with small prime factors
- ▶ Enforce $0 \pmod{p_i}$ for each prime factor p_i of m

Hamiltonian Cycles: Flinders HCP Challenge Graphs

Evaluation on reasonably large instances from the Flinders HCP Challenge Graphs suite

- ▶ Runtime (s) of CaDiCaL on binary adder and LFSR
- ▶ Smallest k such that 2^k (or $2^k - 1$) is larger than $|V|$

graph #	$ V $	$ E $	adder (2^k)	LFSR ($2^k - 1$)
424	2466	4240	> 3600	> 3600
446	2557	4368	> 3600	> 3600
470	2740	4509	2500.61	> 3600
491	2844	4267	173.46	245.92
506	2964	4447	78.29	244.48
522	3060	4591	84.51	611.46
526	3108	4663	160.73	544.97
529	3132	4699	69.69	275.13

Hamiltonian Cycles: Chinese Remainder Results

Evaluation with CaDiCaL on various cycle lengths

✗ : First solution consists of multiple cycles

✓ : First solution consists of a single cycle

graph #	2	6	12	60	105	420
424	9.81 ✗	665.18 ✗	340.11 ✗	307.71 ✗	494.11 ✓	488.70 ✓
446	13.24 ✗	334.62 ✗	169.52 ✗	380.47 ✗	573.38 ✓	722.23 ✓
470	17.08 ✗	166.16 ✗	152.31 ✗	933.36 ✗	501.91 ✗	840.89 ✓
491	0.06 ✗	22.04 ✗	7.47 ✓	34.45 ✓	123.36 ✓	135.22 ✓
506	0.11 ✗	31.75 ✗	19.24 ✓	33.48 ✓	28.73 ✓	63.20 ✓
522	0.63 ✗	5.66 ✗	32.95 ✓	133.40 ✓	30.40 ✓	67.03 ✓
526	0.05 ✗	24.16 ✗	71.67 ✓	34.37 ✓	34.69 ✗	158.69 ✓
529	0.40 ✗	17.90 ✗	60.19 ✓	48.09 ✓	42.33 ✓	365.58 ✓

Lazy Encodings: Beyond Propositional Logic

What if our formula looks like this?

$$(p \wedge \bar{q} \vee a = f(b - c)) \wedge (g(b) \neq c \vee a - c \leq 7)$$

Talks about integers, functions, sets, lists, ...

We can transform it into a SAT formula

- ▶ can only find solutions within bounds
- ▶ very inefficient, so bounds are small

Better idea: combine SAT with special solvers for theories

Lazy Encodings: Satisfiability Modulo Theories (SMT)

Equality and Uninterpreted Functions

EUf = $\langle f, g, h, \dots, =, \text{axioms of equality \& congruence} \rangle$

Linear Integer Arithmetic

LIA = $\langle 0, 1, \dots, +, -, =, \leq, \text{axioms of arithmetic} \rangle$

Arrays, Strings, bitvectors, datatypes, quantifiers, ...

Theories can be combined!

Lazy Encodings: SMT Solvers

- ▶ Z3 (Microsoft): <https://github.com/Z3Prover/z3/wiki>
- ▶ CVC4 (Stanford): <http://cvc4.cs.stanford.edu/web/>
- ▶ Yices (SRI): <http://yices.csl.sri.com/>
- ▶ Boolector (JKU Austria): <https://boolector.github.io/>

Next lecture we will go over SAT and SMT solvers in practice!

Representations for Automated Reasoning

Marijn Heule

**Carnegie
Mellon
University**

<http://www.cs.cmu.edu/~mheule/15816-f21/>

Automated Reasoning and Satisfiability

September 15, 2020